

Consistent perturbations in an imperfect fluid

Ignacy Sawicki,^a Ippocratis D. Saltas,^b Luca Amendola^a and Martin Kunz^c

^aInstitut für Theoretische Physik, Ruprecht-Karls-Universität Heidelberg
Philosophenweg 16, 69120 Heidelberg, Germany

^bAstronomy Centre, University of Sussex
Falmer, Brighton, BN1, 9QH, United Kingdom

^cDépartement de Physique Théorique and Center for Astroparticle Physics
Université de Genève
Quai E. Ansermet 24, CH-1211 Genève 4, Switzerland

E-mail: ignacy.sawicki@uni-heidelberg.de, i.saltas@sussex.ac.uk,
l.amendola@thphys.uni-heidelberg.de, martin.kunz@unige.ch

Abstract. We present a new prescription for analysing cosmological perturbations in a more-general class of scalar-field dark-energy models where the energy-momentum tensor has an imperfect-fluid form. This class includes Brans-Dicke models, $f(R)$ gravity, theories with kinetic gravity braiding and generalised galileons. We employ the intuitive language of fluids, allowing us to explicitly maintain a dependence on physical and potentially measurable properties. We demonstrate that hydrodynamics is not always a valid description for describing cosmological perturbations in *general* scalar-field theories and present a consistent alternative that nonetheless utilises the fluid language.

We apply this approach explicitly to a worked example: *k-essence* non-minimally coupled to gravity. This is the simplest case which captures the essential new features of these imperfect-fluid models. We demonstrate the generic existence of a new scale separating regimes where the fluid is perfect and imperfect. We obtain the equations for the evolution of dark-energy density perturbations in both these regimes. The model also features two other known scales: the Compton scale related to the breaking of shift symmetry and the Jeans scale which we show is determined by the speed of propagation of small scalar-field perturbations, i.e. causality, as opposed to the frequently used definition of the ratio of the pressure and energy-density perturbations.

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1 Introduction

Many models, better or worse motivated by fundamental physics, have been proposed as an alternative to the cosmological constant for providing a mechanism to accelerate the expansion of the universe at late times. Perhaps the simplest, and therefore most studied, are models which feature a single extra scalar degree of freedom, which is treated as classical. The archetypal example of this dark energy (“DE”) are quintessence models [1, 2], where a canonical scalar field rolls in a potential resulting a time-varying equation of state and allowing for (small) inhomogeneities in the dark-energy fluid. On the other hand, $f(R)$ gravity models [3], where the Einstein-Hilbert action for gravity is modified, allow for a different response of the metric to the presence of matter from that in general relativity (“GR”) and usually are discussed as examples of modified gravity. However, these models can also be reformulated as a particular subclass of Brans-Dicke theories featuring a single extra scalar, albeit non-minimally coupled to gravity [4, 5]. It was quickly understood that whereas the background expansion history can be indistinguishable between various classes of these models, the evolution of dark-matter perturbations can differ significantly and therefore formation of large-scale structure provides the key to understanding the role played by these theories, if any, in accelerating the universe [6].

k-essence was introduced as an extension of quintessence models to non-canonical kinetic terms [7–9] and can be interpreted as a perfect fluid with a speed of sound different from that of light [10]. This allows for a Jeans scale smaller than the cosmological horizon and therefore

for DE which can cluster significantly. In particular, the discovery of generic attractors where the equation of state approaches that of the vacuum while the sound speed tends to zero [11] also spurred the discussion of such models as describing both dark energy and dark matter through a single degree of freedom [12–14]. One tends to think of these dark-energy models as perfect fluids, obeying hydrodynamics and studies the evolution of linear perturbations by considering the conservation of the perturbed energy-momentum tensor [15–17].

On the other hand, when studying large-scale structure in $f(R)$ theories [18, 19], one tends to think in terms of a modification of the response of the two scalar gravitational potentials, Φ and Ψ , to matter perturbations. In particular the generic result for this class of theories is that the lensing potential, $\Phi - \Psi$, is “unaffected” and depends only on the matter perturbations and an evolving Newton’s constant, while the modification of gravity creates anisotropic stress giving $2\Phi + \Psi = 0$ ¹. These relationships are simple and therefore are frequently generalised in order to constrain such theories from data. A large number of phenomenological parameterisations of models of modified gravity exist [6, 21–28] and usually focus on directly modifying the relationships between the potentials and matter perturbations without necessarily asking what this implies for the underlying model.² Considering a lack of well-motivated models, one may argue that this is not a significant problem.

$f(R)$ models and this sort of modifications of gravity are quite constrained by Solar-System tests as a result of their being equivalent to fifth forces. One can find viable models [30–32] which can evade the constraints through the chameleon effect [33, 34], or one can propose that the extra scalar does not couple to baryons, but only to dark matter [35, 36]. These types of interacting DE models can have dynamics on large scales very similar to modified gravity models, since the contribution of baryons to the energy density is very subdominant. However, philosophically they are no longer “modified gravity” since the coupling is not universal — it no longer couples purely to the energy-momentum tensor. On the other hand, the full breadth of such models is much wider than for $f(R)$ theories, which are but a subclass. When studying interacting dark energy, one tends to treat the scalar field differently from matter: the equation of motion for the scalar is usually parameterised and solved, while the dark matter and baryons are treated as a hydrodynamical fluid.

Another class of scalar-field models was discovered by considering gravity modifications arising from a potential higher-dimensional embedding of our universe [37–39]. In the decoupling limit [40–42], this DGP gravity reduces to a scalar theory, where the scalar Lagrangian contains second derivatives of the scalar field and therefore is not in the *k-essence* class. A further generalisation of such scalar-field theories featuring second derivatives was then obtained in refs [43, 44] and named the *galileon*. Such theories are not consistent on general curved backgrounds, and the appropriate covariantisation was obtained in [45, 46]. Kinetic gravity braiding, a generalisation of the DGP decoupling limit, was independently found in refs [47, 48] and soon it was proven that the most general theory of a single scalar with no more than second derivatives is a similar generalisation of the galileons [49]. Indeed these generalised galileons were discovered a long time ago [50] and largely ignored. The higher-dimensional origin of the galileon theories was studied in refs [51–54], while the galileon theories also have appeared in the decoupling limit of consistent massive-gravity as actions for the helicity-0 mode of the massive graviton [55–57].

¹For an investigation of the relation between anisotropic stress and cosmological evolution in non-linear gravity models see ref [20].

²Although see ref. [29] where a very general parameterisation is given based on symmetries of the Einstein equations.

These theories all contain second derivatives in the action and therefore have energy-momentum tensors (“EMT”) which do *not* have the form of a perfect fluid [58, 59] and therefore provide for new phenomenology. For example, the null-energy condition can be violated stably, resulting in explicitly ghost-free phantom DE models or even permitting the construction of initial phases of evolution of the universe alternative to inflation [60–62]. Equations for linear cosmological perturbations for the most general of such theories were derived in [63].

What strikes in the above discussion is the multitude of approaches that are used to study perturbations, each chosen depending on the underlying Lagrangian for the scalar. However, all the models discussed here are structurally extremely similar. A common framework for describing perturbations would allow for a single consistent method of constraining the class of models of DE comprising a single scalar. One could argue that solving the equation of motion for the scalar provides exactly such a framework. However, if we assume that the DE cannot be studied directly, but only through its effect on the gravitational field in which baryons and light propagate, the equation of motion contains too much information. Gravity couples only to the EMT and therefore will mostly depend on the fluid properties of the DE and not on the precise value of the scalar field.

The case of *k-essence* is a pertinent example. Any particular choice within this class of Lagrangians gives a particular expansion history for the cosmological background, with a particular evolution of the equation-of-state parameter, w . This solution (based on the choice of Lagrangian) then also determines the speed of propagation of the scalar-field perturbations, c_s^2 . Given the history of these two variables, the evolution of linear energy-density perturbations of a *k-essence* DE is *completely* determined at all scales. Any *k-essence* model which happens to have the same history of evolution of w and c_s^2 will have exactly the same evolution of linear perturbations, even if the two Lagrangians are very different and only happen to have coincided in this way on two individual solutions. There is therefore no way to reconstruct a Lagrangian even limited to this single class of theories, even if we ignore the fact that the data that can be obtained will only ever be limited to some maximal precision and we will only be able to measure the properties of the DE indirectly. Unless DE interacts directly with baryonic matter in a way that is not equivalent to redefining a gravitational metric, we will only be able to measure the properties of the DE perturbations indirectly by looking at scale-dependent modifications of the DM power spectrum. Given this, the best we can hope for is to measure w through its effect on geometry and the sound speed of DE by detecting some sort of scale of transition in behaviour that would be consistent with a Jeans length [64, 65].

A similar situation occurs when dark energy is described by the class of $f(R)$ theories. Here, there appears to be only one scale, that of the (time-varying) Compton wavelength of the scalar, the chameleon effect allowing for the restoration of GR in dense regions through non-linearities notwithstanding. On both sides of this scale, the evolution of linear perturbations is completely determined by the fact that we have chosen to consider this limited class of Lagrangians, just as it was for *k-essence*. Indeed, specifying the evolution of w and of the Compton scale gives us the totality of the information that can be gleaned from the background and the linear perturbations.

In fact, the standard discussion of $f(R)$ gravity neglects two more scales which are implicitly contained in the theory. Firstly, there is still the Jeans scale. However, in this case, the speed of sound is equal to that of light and therefore the Jeans scale lies at the

cosmological horizon and is thus unobservable. Secondly, there exists a new scale controlling whether the fluid is perfect or imperfect, which will be the subject of much of this paper. $f(R)$ models are a limit where this scale lies at infinity and therefore it is also unobservable there.

The main aim of this paper is to provide a prescription for describing linear perturbations which as far as possible depends only on physical properties of the model, such as the equation of state or the sound speed, which one could hope to measure. Our approach is to describe the DE through the conservation equations for its EMT, rather than the equation of motion. This provides the evolution equations for the energy-density perturbations which are then directly connected to the gravitational potentials. For any theory, these equations can be solved provided we supply two closure relations: relationships of the pressure perturbation and anisotropic stress to the energy-density perturbation. The solutions for linear perturbations of any fluid are always fully determined by these two functions.³ The question we answer is how to properly obtain these relationships for a model featuring a single scalar degree of freedom.

We show explicitly that even for general *k-essence* models these closure relations, which are well known [66], are not hydrodynamical and one should never a priori assume hydrodynamical behaviour in more general cases. However, on any time slice all components of the EMT are determined by the values of the degrees of freedom at that moment in time. Thus, for any class of models based on degrees of freedom with classical equations of motion, we can obtain such closure relations. Indeed these relations define the totality of the properties of the dark energy at the linear level of perturbations. We provide a form of these closure relations for any scalar-field theory and we explicitly calculate them for *k-essence* coupled non-minimally to gravity. This particular class contains both *k-essence* and $f(R)$ theories and as a result of the non-minimal coupling to gravity features second-derivatives of the scalar in the EMT in the Jordan frame. In this sense, it mimics the form of the EMT possessed by more-general galileon theories and therefore is a useful example.

We show that in general scalar-field models we have three scales: the Jeans scale, the Compton scale and a new scale determining whether the fluid is in a perfect or imperfect regime. This is *in addition* to the cosmological-horizon scale. These three scales are essentially independent and can lie in any order and are determined by the Lagrangian class and the particular background solution. This new transition scale appears in all scalar-field models more general than *k-essence* and is determined by the relative importance of the second-derivative terms and the *k-essence* terms in the perturbed EMT. For example, it is only inside this scale that the DE fluid can carry anisotropic stress at all. This scale is not visible in the equation of motion for the scalar, but only appears when the EMT is considered.

As an example of the power of our prescription, we obtain the equations for the evolution of energy-density perturbations for the non-minimally coupled *k-essence* DE model and solve them analytically under the assumption that the background evolution exhibits scaling behaviour (i.e. has a constant equation-of-state parameter).

The paper is structured as follows: we start in section 2 by describing a formalism for decomposing the EMT into the fluid variables covariantly. This description allows us to obtain exact forms for, among others, the energy density and pressure in terms of the

³Some EMT non-conservation issues related to the evolution of the effective Planck mass notwithstanding.

scalar-field, which can then be perturbed directly. In this way, we obtain the conservation equations for the linear perturbations of the EMT in section 2.2. In section 2.3 we discuss what hydrodynamics implies for the relationship between pressure and energy density in perfect fluids, and by deriving this relationship for general *k-essence* models in section 2.4 we show that the hydrodynamical relations are not obeyed. We then propose a form for these closure relations which would be obeyed by any DE model comprising a single scalar field in section 2.5. In section 3, we turn to a non-minimally coupled *k-essence* as worked example. We discuss the general properties of this theory, before turning to calculate the closure relations between the pressure, energy density and anisotropy perturbations in section 3.3. We demonstrate explicitly how to calculate them and show that they are of the form contained within the general parameterisation introduced earlier. We explicitly demonstrate the existence of the three scales in the problem and derive the equation for the evolution of the density contrast in these models. We summarise our findings in section 4.

2 Fluid formalism

It is well known that theories such as *k-essence* can be much more intuitively described in terms of relativistic hydrodynamics [7]. We are going to employ this language extensively and show that it provides a very natural framework in which to discuss theories with contributions to the energy-momentum tensor containing second derivatives. This language was also previously employed to study scalar field theories with kinetic gravity braiding [58] (to where we refer the reader for a more detailed exposition of this formalism), but was developed for relativistic potential flows much earlier [67–70]. We will proceed in this manner, leaving aside for the moment the question of whether the more general scalar-field theories we would like to describe are in fact hydrodynamical fluids.

2.1 Standard definitions

In this discussion, we are going to be describing a scalar-field theory. Following standard practice, we will define a canonical kinetic term for the scalar,⁴

$$X \equiv -\frac{1}{2}\partial_\mu\phi\partial^\mu\phi. \quad (2.1)$$

In cosmology one usually takes gradients as being time-like. It is only with this choice that the fluid description can be valid, thus we will always assume that $X > 0$. The scalar-field gradient now gives a time-like vector field, which when appropriately normalised can be identified with a velocity field,

$$u_\mu \equiv -\frac{\partial_\mu\phi}{\sqrt{2X}}. \quad (2.2)$$

This velocity field is usually referred to as the fluid rest-frame, but in the case of the more general models we discuss, this is not always the case: there remains a non-vanishing energy flow in the frame moving with the velocity u^μ . We will therefore call the frame defined by (2.2) the *scalar frame*. We comment further on this frame choice in Appendix B.

Given the velocity field, we can define a derivative along u^μ (a material derivative)

$$u^\mu\nabla_\mu = \frac{d}{d\tau}, \quad (2.3)$$

⁴We employ the metric signature convention $(-+++)$.

thus making τ the proper time of an observer comoving with the scalar frame. We will assume that this time derivative be positive, and therefore u^μ be future directed. This allows us to think of ϕ as a clock.

We can then proceed and define the transverse projector,

$$\perp_{\mu\nu} = g_{\mu\nu} + u_\mu u_\nu, \quad (2.4)$$

which plays the role of the first fundamental form in the hypersurfaces $\Sigma_\phi : \phi(x) = \text{const}$, i.e. the spatial metric for the scalar-frame observer. The projector (2.4) allows us to decompose vectors and gradients into time and spatial parts as observed in the scalar frame,

$$\nabla_\mu = -u_\mu u^\lambda \nabla_\lambda + \perp_\mu^\lambda \nabla_\lambda = -u_\mu \frac{d}{d\tau} + \overline{\nabla}_\mu. \quad (2.5)$$

We will refer to $\overline{\nabla}_\mu$ as the spatial derivative. We can then proceed to decompose the gradient of the vector field u^μ , by defining the acceleration, a spatial vector field

$$a_\mu \equiv \frac{d}{d\tau} u_\mu = u^\lambda \nabla_\lambda u_\mu, \quad (2.6)$$

which can also be expressed in terms of the scalar field

$$a_\mu = -\perp_\mu^\lambda \nabla_\lambda \ln \frac{d\phi}{d\tau}, \quad (2.7)$$

and performing the standard kinematical decomposition

$$\nabla_\mu u_\nu = -u_\mu a_\nu + \sigma_{\mu\nu} + \frac{1}{3} \perp_{\mu\nu} \theta, \quad (2.8)$$

where $\theta \equiv \nabla_\mu u^\mu$ is the expansion and

$$\sigma_{\mu\nu} \equiv \perp_{(\mu}^\lambda \nabla_{\lambda} u_{\nu)} - \frac{1}{3} \perp_{\mu\nu} \theta \quad (2.9)$$

is the fully spatial (transverse) and traceless symmetric shear tensor. The rotation tensor (the antisymmetric spatial part of (2.8)) vanishes by virtue of the Frobenius theorem, since u_μ is a gradient of a scalar.

In ref. [58], it was proposed that instead of the canonical scalar kinetic term, X , a more physical variable is in fact

$$m \equiv \frac{d\phi}{d\tau} = u^\mu \nabla_\mu \phi. \quad (2.10)$$

On the cosmological background, which will be the focus of our discussion, the variable m reduces to the derivative of the scalar with respect to the coordinate time, $d\phi/dt$. It will also be helpful to give the decomposition of the second derivative in terms of the kinematical variables:

$$\nabla_\mu \nabla_\nu \phi = -\frac{dm}{d\tau} u_\mu u_\nu - m \left(a_\mu u_\nu + u_\mu a_\nu - \sigma_{\mu\nu} - \frac{1}{3} \perp_{\mu\nu} \theta \right). \quad (2.11)$$

In the hydrodynamical picture, the variable m can be interpreted as playing the role of the chemical potential for the charges which make up the fluid. In this picture, a scalar-field theory is a thermodynamical system at zero temperature, with m acting as a chemical potential and any dependence on ϕ representing an explicit dependence of the properties on

the internal clock. As argued in [71–73], scalar-field theories such as *k-essence* are perfect hydrodynamical fluids *only* when they are shift symmetric, with the action invariant under the transformation $\phi(x) \rightarrow \phi(x) + \text{const}$, i.e. hydrodynamics is only valid when there is no explicit dependence on the clock carried by the fluid. We will return to this point in the discussion of the validity of hydrodynamics in sections 2.3 and 2.4.

The configuration of a minimally coupled *k-essence* theory is fully specified by the pair (ϕ, m) . Theories with second derivatives in the Lagrangian, such as kinetic gravity braiding [58] are somewhat more complicated, since their configuration can also depend on quantities such as the expansion θ . Typically, the energy momentum tensor of these theories will contain second derivatives of the scalar. We will discuss this sort of example in section 3.

2.2 Evolution equations from EMT conservation

At this stage, it is useful to jump ahead and be a little more specific. In scalar models coupled non-minimally to gravity, the fundamental Planck mass, M_{Pl} will in general be modified by a function of the scalar, let us call it \varkappa , through a coupling $e^{\varkappa}R$ in the Lagrangian. This would mean that the energy momentum tensor (“EMT”) for the DE would in general contain a term proportional to the Einstein tensor, $(1 - e^{\varkappa})G_{\mu\nu}$. When the Einstein tensor is solved for through the Einstein equations, the energy-momentum tensor for the dark energy depends explicitly on the dark-matter configuration, which we would like to think of as a separate degree of freedom with its own dynamics (in fact, potentially multiple degrees of freedom: dark matter, baryons, radiation, etc.). The prescription we introduce in this paper calls for separating out this dependence, so as to leave the DE EMT as much as possible independent of the particular configuration of the external matter. We will therefore explicitly solve for the Einstein tensor and introduce an EMT, $T_{\mu\nu}^X$ with the Einstein tensor removed. The Einstein equations are then given by

$$G_{\mu\nu} = T_{\mu\nu}^X + e^{-\varkappa}T_{\mu\nu}^{\text{ext}}, \quad (2.12)$$

where we have chosen units where $M_{\text{Pl}}^2 = 8\pi G_{\text{N}} = 1$. $T_{\mu\nu}^{\text{ext}}$ is the possible EMT external to the dark-energy system, for example representing dark matter, which would be explicitly conserved, $\nabla^\mu T_{\mu\nu}^{\text{ext}} = 0$. All the fluid variables relating to the external matter will appear suppressed by $e^{-\varkappa}$. The price for such a definition is the fact that $T_{\mu\nu}^X$ is not conserved in the presence of matter external to the system,

$$\nabla^\mu T_{\mu\nu}^X = e^{-\varkappa}T_{\mu\nu}^{\text{ext}}\nabla^\mu \varkappa. \quad (2.13)$$

In this paper, we will consider the isolated dark-energy system, setting $T_{\mu\nu}^{\text{ext}} = 0$, only making comments in passing when the presence of external matter modifies significantly our results. We will return to study the combined dark-energy-dark-matter system in another work.

A natural question to ask is why we are not performing the analysis in the Einstein frame, which is the typical way in which non-minimally coupled theories are usually discussed [74]. Indeed for the model we present in the worked example in section 3, such a transformation can be performed and the analysis of such a setup is standard: in the Einstein frame we are dealing with a normal perfect fluid. However, more general models of DE (kinetic gravity braiding, generalised galileons) feature an intrinsic deviation in the EMT from the perfect-fluid form which cannot be removed by a redefinition of frame. The Einstein frame defined as the frame where the kinetic terms of the metric and of the scalar field are unmixed, does not necessarily exist in more general models. Our motivation is that the

simple model presented in our worked example, in the Jordan frame, has many features in common with the general galileon models and therefore our prescription as employed in the Jordan frame will transfer directly to more complex situations.

In general, the baryons, DM and gravity could all have different frames in which they have minimal couplings. We can always pick one of those frames as the basis for the analysis. Since most of our observations are to do with baryons, it simplifies matters to assume that they move on geodesics of some metric. This is the metric in which they are minimally coupled. But in this frame, the DM and gravity could both have some sort potentially different non-minimal coupling for the scalar.

Once we have defined a velocity field, we can decompose the EMT into the fluid variables that will be seen by the observer \mathcal{O} comoving with velocity u^μ :

$$T_{\mu\nu}^X = \mathcal{E} u_\mu u_\nu + \perp_{\mu\nu} \mathcal{P} + u_\mu q_\nu + u_\nu q_\mu + \tau_{\mu\nu}, \quad (2.14)$$

where $\mathcal{E} \equiv T_{\mu\nu}^X u^\mu u^\nu$ is the energy density observed by \mathcal{O} , $\mathcal{P} \equiv \perp^{\mu\nu} T_{\mu\nu}^X / 3$ is the pressure, $q_\lambda \equiv -T_{\mu\nu}^X \perp_\lambda^\mu u^\nu$ is the energy flow as seen by \mathcal{O} and $\tau_{\lambda\rho} \equiv T_{\mu\nu}^X \left(\perp_\lambda^\mu \perp_\rho^\nu - \frac{1}{3} \perp_{\lambda\rho} \perp^{\mu\nu} \right)$ is the spatial and traceless viscous-stress tensor.⁵ The observed values of the fluid quantities are not independent of the choice of observer and are not just gauge effects. We discuss this in Appendix B. As implied by the decomposition of the second derivative of a scalar eq. (2.11), EMTs containing second derivatives will in general have contributions to all the components of the EMT, implying that the EMT has imperfect-fluid form: it is impossible to find a velocity frame in which it could be decomposed and possess a perfect-fluid form.

At least for some models (in particular, the one we discuss as a worked example in section 3), we can rewrite the energy flow q_λ and viscous stress $\tau_{\mu\nu}$ in terms of a scalar potential with an appropriate number of derivatives:

$$\begin{aligned} q_\lambda &= \perp_\lambda^\mu \nabla_\mu q, \\ \tau_{\lambda\rho} &= \left(\perp_\lambda^\mu \perp_\rho^\nu - \frac{1}{3} \perp_{\lambda\rho} \perp^{\mu\nu} \right) \nabla_\mu \nabla_\nu \pi \end{aligned} \quad (2.15)$$

with q the potential for energy flow and π the anisotropic stress.

We present a detailed discussion of the conservation equations for the EMT using the language of covariant decomposition in Appendix A. Here we are just going to discuss the linear scalar perturbations on a spatially flat Friedmann-Lemaître-Robertson-Walker (“FLRW”) metric working in the Newtonian gauge, assuming that vector and tensor perturbations be negligible,

$$ds^2 = -(1 + 2\Psi)dt^2 + a^2(t)(1 + 2\Phi)d\mathbf{x}^2. \quad (2.16)$$

The exact equations presented in the appendix, (A.8) and (A.9), can be perturbed directly. Doing so, we obtain the standard results. At the background level only the timelike equation contributes, giving the energy-conservation equation,

$$\dot{\mathcal{E}} + 3H(\mathcal{E} + \mathcal{P}) = 0, \quad (2.17)$$

⁵The quantity q_λ is usually called the heat flow. However, as was demonstrated in [58] for kinetic gravity braiding theories, entropy production and heat flow are zero while q_λ does not vanish. In that case, the vector q_λ quantifies the energy carried by charges moving in the scalar frame.

with the overdot representing the derivative with respect to coordinate time t . At the linear level in perturbations, we can transform into momentum space obtaining:

$$\delta\dot{\mathcal{E}} + 3H(\delta\mathcal{E} + \delta\mathcal{P}) + (\mathcal{E} + \mathcal{P})(\Theta + \dot{\Phi}) - \frac{k^2\delta q}{a^2} + \dot{q}\Theta = 0, \quad (2.18)$$

$$\begin{aligned} &(\mathcal{E} + \mathcal{P})\left(\dot{\Theta} + 2H\Theta - \frac{k^2}{a^2}\Psi\right) - \frac{k^2}{a^2}\delta\mathcal{P} + \Theta\dot{\mathcal{P}} + \\ &- 3H\left(\frac{k^2}{a^2}\delta q - \dot{q}\Theta\right) - \frac{1}{a^2}\left(k^2\delta q - a^2\dot{q}\Theta\right)' - \frac{2}{3}\frac{k^4}{a^4}\delta\pi = 0, \end{aligned} \quad (2.19)$$

where the δ prefix signifies a perturbed quantity and we have defined the divergence of the perturbation of spatial velocity for the observer in the scalar frame:⁶

$$\Theta \equiv ik_i\delta u^i = \frac{k^2\delta\phi}{ma^2}. \quad (2.20)$$

It is customary to make the equations (2.18) and (2.19) dimensionless by dividing through by the background energy density: for us, it will prove more convenient to keep them in this form, since we do not wish to at this stage follow the usual practice of defining a sound speed, etc. and we wish to avoid any potential singularities this procedure might introduce in the variables.

The form of the equations (2.18) and (2.19) is in fact gauge invariant, since in the Newtonian gauge eq. (2.16) the metric potentials correspond to the gauge invariant Bardeen potentials [15]. Gauge invariance, however, does not imply frame invariance. The standard references (e.g. ref. [16]) perform the decomposition of the EMT eq. (2.14) in the rest frame of the fluid and therefore do not contain the energy flow terms involving the potential q . The difference between any two frames is a Lorentz boost between the two velocities of the observers in whose rest frame the EMT is decomposed. Given that the background is FLRW, there is only one reasonable frame on the background level, that of the background comoving observer. Thus all the frames differ purely on the level of perturbations, with the boosts transforming equations (2.18) and (2.19) by trading the divergence of energy flow $k^2\delta q$ for a redefinition of the divergence of the momentum perturbation $(\mathcal{E} + \mathcal{P})\Theta$.

For a model with second derivatives in the EMT, the choice of any of these related frames is perfectly fine *apart* from the rest frame. In vacuum configurations, $\mathcal{E} + \mathcal{P} = 0$ and the divergence of momentum perturbation $(\mathcal{E} + \mathcal{P})\Theta$ vanishes. Therefore the rapidity of the boost required to compensate a divergence of energy flow $k^2\delta q$ tends to infinity as we approach the vacuum equation of state. This means that close to vacuum configurations, perturbation theory in terms of the underlying scalar-field variables itself breaks down if we require the frame to be a rest frame.

We have provided more detail for this discussion in Appendix B. Suffice it to say that the presence of the energy flow potential q in the perturbation equations (2.18) and (2.19) will be the origin of much of the novel behaviour described in this paper. Scalar-field theories more general than *k-essence* always contain such a term in the EMT decomposed in the scalar frame (2.2). Close enough to vacuum configurations, the coefficients of the Θ terms in

⁶Note that there is a relative factor of a between Θ defined above and the variable θ used by Ma and Bertschinger [16, Eq. (23b)], so that $\theta_{\text{MB}} = a\Theta$. This is a result of our using coordinate instead of conformal time.

(2.18) and (2.19) driving the evolution of perturbations under normal circumstances vanish and that role is taken over by the q terms. This then allows for such behaviour as a stable violation of the null-energy condition [47, 62] or “phantom crossing”. The existence of the additional derivatives on the δq perturbations compared to the Θ terms will mean that the transition between the two behaviours will not only depend on the equation of state of the dark energy but also on scale.

Another aspect worth noting here is that the background value of q appears in eqs (2.18) and (2.19). This does not have to vanish, only its spatial gradient must at the background level (the total value of q is some function of the scalar field, ϕ and m , so its background value is going to be that function of the background values of ϕ and m).

The aim of this discussion is to reduce the perturbation equations (2.18) and (2.19) into a single evolution equation for the density perturbation $\delta\mathcal{E}$. In the rest frame, this is simple since one can just eliminate the variable Θ . In the scalar frame, in general both Θ and δq have to be eliminated. Luckily, one can exploit the structure of the conservation equations arising from their transformations under Lorentz boosts of u^μ to make an *algebraic* redefinition of Θ to define the divergence of the total energy flow,

$$\Xi \equiv (\mathcal{E} + \mathcal{P})\Theta - \frac{k^2\delta q}{a^2} + \dot{q}\Theta. \quad (2.21)$$

Ξ is indeed proportional to the velocity divergence in the rest frame, Θ_{LL} as defined in eq. (B.6). However, since it is only an algebraic redefinition for us, the meaning of the perturbation variables remains: $\delta\mathcal{E}$, $\delta\mathcal{P}$, etc. are still those measured in the scalar frame and *not* in the rest frame. Importantly, Ξ is not singular when $w = -1$. This definition allows us to transform perturbation equations (2.18) and (2.19) into a simpler set,

$$\delta\mathcal{E} + 3H(\delta\mathcal{E} + \delta\mathcal{P}) + \Xi + (\mathcal{E} + \mathcal{P})\dot{\Phi} = 0, \quad (2.22)$$

$$\dot{\Xi} + 5H\Xi + (\mathcal{E} + \mathcal{P})\frac{\partial^2\Psi}{a^2} - \frac{k^2}{a^2}\delta\mathcal{P} - \frac{2}{3}\frac{k^4}{a^4}\delta\pi = 0. \quad (2.23)$$

The above then need to be furnished with Einstein’s equations. We obtain the independent equations by projecting out particular parts of the Einstein tensor and then perturbing,

$$\delta(G_{\mu\nu}u^\mu u^\nu) = 2\frac{k^2\Phi}{a^2} + 6H\dot{\Phi} - 6H^2\Psi = \delta\mathcal{E}, \quad (2.24)$$

$$2\frac{k^2}{a^2}(\dot{\Phi} - H\Psi) = -\Xi, \quad (2.25)$$

$$\Phi + \Psi = \delta\pi, \quad (2.26)$$

where eq. (2.25) makes the meaning of Ξ as the total energy flow explicit. Contributions from external matter would need to be added to both the sets of equations above.

In order to solve for the evolution of the energy-density perturbations, we need two extra ingredients: the closure relations between $\delta\mathcal{P}$, $\delta\pi$ and $\delta\mathcal{E}$. We shall first discuss these relations in perfect-fluid hydrodynamics and compare them with *k-essence* theories. We show that even for a canonical scalar theory with a potential the standard relation between $\delta\mathcal{P}$ and $\delta\mathcal{E}$ is *not* hydrodynamical, despite the EMT having perfect-fluid form. This departure from perfect-fluid hydrodynamics becomes only worse for more-general scalar-field models. Since all EMTs obey equations (2.22) and (2.23) at linear level, the closure relations provide

the totality of information that can be accessed from gravitational interactions at the linear level.

On the other hand, since we are investigating theories which are described by classical equations of motion, it is possible to obtain the exact closure relations. As we are dealing with equations of motion for a scalar field of second order, we can always express the closure relations in terms of variables defined on a constant-time slice. In section 2.5, we will present a general parameterisation which will serve us as a book-keeping device to ease these calculations for particular classes of models. In section 3 we will take a specific model and perform this calculation explicitly. Our aim will be to obtain closure relations that to the largest extent possible depend on a set of physical parameters of the theory, such as the equation of state or the speed of sound.

2.3 Perfect fluids and hydrodynamics

In this section we will review the properties of perfect fluids. For a more detailed discussion, we recommend the review [75]. The perfect-fluid EMT is obtained by averaging a distribution of particles to obtain fluid elements and takes the form

$$T_{\mu\nu} = \mathcal{E}u_\mu u_\nu + \mathcal{P} \perp_{\mu\nu}, \quad (2.27)$$

with the energy density and pressure the only fluid variables. In Newtonian hydrodynamics one starts from the axioms of energy and momentum conservation, with the separate requirement that mass be conserved. From these the conservation of the EMT is obtained. This has to be appropriately generalised for relativistic fluids. The energy and momentum conservation equations become the continuity and Euler equations, while mass is no longer conserved. This axiom needs to be generalised to some appropriate charge conservation.

The velocity field u_μ defines the rest frame (or the lagrangian observer) for the fluid elements. This velocity must be by definition time-like since in the construction of the EMT it is the tangent to the world-line of individual fluid elements. It is of course possible to observe this EMT in a different frame. However, it will then not in general have a perfect-fluid form and the measured energy density and pressure will be appropriately transformed. We discuss the issue of frame choice in detail in appendix B. In the rest frame, we can relate the pressure to the energy density through

$$\mathcal{P} = \mathcal{P}(\mathcal{E}, S), \quad (2.28)$$

where S is the entropy. In cosmology one often considers either adiabatic or entropy perturbations. The former are such perturbations of the configuration of the fluid that the pressure and energy-density perturbations are proportional to each other. The latter are such that despite a non-vanishing pressure perturbation, the energy density is not affected, i.e. these are perturbations only of composition of the fluid assuming constant energy density. The perturbations can be written as

$$\delta\mathcal{P} = c_s^2 \delta\mathcal{E} + \sigma \delta S \quad (2.29)$$

where we have *defined* the speed of sound c_s^2 as the coefficient in this equation. This sound speed then determines such physical properties as the Jeans length. As we have shown in appendix B, linear frame transformations affect the fluid variables at second order and therefore this form is independent of the choice of frame perturbation.

For the purpose of calculating the anisotropies in the cosmic microwave background (“CMB”), when the photons and baryons are strongly coupled and in equilibrium, one can

think of the photon-baryon mixture as making up one fluid. The entropy perturbations have a clear meaning of being a perturbation in the relative densities of photons and baryons while keeping the energy density constant. In such a case, the expression (2.29) is easily calculable [76, pg. 307]. This language is useful since a proof exists that at very large scales, the entropy perturbations are suppressed and only the adiabatic ones contribute [59, 77, 78]. However, in late-time cosmology, we are performing observations inside the Hubble horizon and therefore the entropy perturbations cannot in general be neglected.

As we will show here, this description is not so helpful for classical scalar-field theories when applied to the late universe. The issue boils down to the question of what the degrees of freedom that are being perturbed in order to obtain the expression (2.29) are. There exists another description of fluids, much closer to our classical-scalar-field case, in terms of the thermodynamical potentials, temperature T and chemical potential μ . For a hydrodynamical perfect fluid, it is always possible to rewrite the system (2.28) in terms of these two potentials,

$$\begin{aligned}\mathcal{P} &= \mathcal{P}(T, \mu), \\ \mathcal{E} &= \mathcal{E}(T, \mu).\end{aligned}\tag{2.30}$$

The pressure and energy-density perturbations are now completely determined by the perturbations in the values of T and μ . In general, we now have two types of propagating modes, one corresponding to temperature perturbations, the other to chemical-potential perturbations, i.e. effectively composition.

Let us for the moment simplify and assume that the fluid be adiabatic: such a fluid is dependent on just one thermodynamical potential, e.g. the temperature, $\mathcal{P}(T), \mathcal{E}(T)$. For the avoidance of ambiguity, let us also specify that we are observing this fluid in the rest frame. The perturbations of energy density and pressure in the rest frame are then related by

$$\delta\mathcal{P} = c_s^2 \delta\mathcal{E},\tag{2.31}$$

where $c_s^2 \equiv \mathcal{P}_{,T}/\mathcal{E}_{,T}$ must have the same meaning as the sound speed defined in (2.29). In this language, the temperature is the degree of freedom which determines everything about the configuration of the fluid. Indeed, when discussing the photon gas, one does tend to talk of temperature perturbations rather than energy-density perturbations. For a photon fluid the chemical potential vanishes, since photons carry no charge, and thus everything depends just on the temperature. Perturbing the temperature perturbs the Bose-Einstein occupation function and therefore the number density of photons, the energy density and pressure. From this discussion, it should be clear that the evolution of the total pressure and energy density with respect to a time coordinate in the frame of the observer τ is also just related by the same sound speed,

$$\dot{\mathcal{P}} = c_s^2 \dot{\mathcal{E}},\tag{2.32}$$

since everything is just a function of T and that $c_s^2 = w$, the equation-of-state parameter.

In the non-adiabatic case, when both the temperature and the chemical potential are relevant, there exists another direction in which the system could fluctuate. The pressure perturbation can now be written down as

$$\begin{aligned}\delta\mathcal{P} &= \mathcal{P}_{,T}\delta T + \mathcal{P}_{,\mu}\delta\mu, \\ \delta\mathcal{E} &= \mathcal{E}_{,T}\delta T + \mathcal{E}_{,\mu}\delta\mu.\end{aligned}$$

We can also write down a very similar expression for the time evolution of the total quantities

$$\begin{aligned}\dot{\mathcal{P}} &= \mathcal{P}_{,T}\dot{T} + \mathcal{P}_{,\mu}\dot{\mu}, \\ \dot{\mathcal{E}} &= \mathcal{E}_{,T}\dot{T} + \mathcal{E}_{,\mu}\dot{\mu}.\end{aligned}$$

We can eliminate the δT perturbation by combining these two sets of equations to obtain

$$\delta\mathcal{P} = C^2\delta\mathcal{E} + (\dot{\mathcal{P}} - C^2\dot{\mathcal{E}})\frac{\delta\mu}{\dot{\mu}} \quad (2.33)$$

with $C^2 \equiv \mathcal{P}_{,T}/\mathcal{E}_{,T}$. We can now compare this expression to eq. (2.29) and see that they have the same form. We can read off the definition of the sound speed in terms of the thermodynamic potentials and find an expression for the entropy perturbation, in terms of $\delta\mu$. Note that we could have eliminated the chemical potential instead of the temperature perturbation. It is important to stress that the form of the expression (2.33) is independent of the frame of the observer, provided that it differs from the rest frame only at linear order. In particular it does not rely on coordinates, but is purely determined by the fluid configuration.

The idea of this rewriting of the system (2.28) in terms of the thermodynamical potentials is that we can rewrite the system in terms of the degrees of freedom that we are free to perturb independently on any spatial hypersurface and then obtain an effective equation (2.33) eliminating some of them. As we will see, in a classical scalar-field theory, we also have such underlying degrees of freedom (the field value and the magnitude of its first derivative) on which all the fluid variables depend. The EMTs of the scalar systems look more similar to eq. (2.30) than to eq. (2.28).

A fundamental assumption that we have used in the above is that no hydrodynamic quantity is an explicit function of time: pressure and energy both depend *only* on the thermodynamical potentials. The potentials evolve according to the laws of thermodynamics, changing the configuration of the fluid, but if the potentials were to be returned to the same values as at some point in the past, the pressure and energy density would also *return* to their previous values. This is why the phase diagram of water does not change over time.

One could imagine violating this principle. This would really mean that what the fluid is changes as a function of time, not just that its configuration evolves. Let us go back to a single thermodynamic potential, T , but this time together with an explicit dependence on rest-frame time τ . We can write the derivative w.r.t. the rest-frame time τ ,⁷

$$\begin{aligned}\frac{d\mathcal{P}}{d\tau} &= \partial_\tau\mathcal{P} + \mathcal{P}_{,T}\frac{dT}{d\tau}, \\ \frac{d\mathcal{E}}{d\tau} &= \partial_\tau\mathcal{E} + \mathcal{E}_{,T}\frac{dT}{d\tau}.\end{aligned} \quad (2.34)$$

At a single point in time, i.e. on some spatial hypersurface in the rest frame, the perturbations will look as they do for the adiabatic fluid, since the only perturbed variable will be T ,

$$\begin{aligned}\delta\mathcal{P}_{\text{rf}} &= \mathcal{P}_{,T}\delta T, \\ \delta\mathcal{E}_{\text{rf}} &= \mathcal{E}_{,T}\delta T.\end{aligned} \quad (2.35)$$

⁷Note that using any other time coordinate here would mix the meaning of what should be considered to be a perturbation of the fluid in the rest frame and a perturbation of the rest frame itself. We should always rewrite all the variables to be in terms of rest-frame quantities where the fluid has a perfect form.

However now, when we transform into the frame of a different observer, the value of the perturbation observed by them is going to be corrected by the fact that the rest-frame time and the proper-time of the two observers are no longer the same,

$$\begin{aligned}\delta\mathcal{P} &= \partial_\tau\mathcal{P}\delta\tau + \delta\mathcal{P}_{\text{rf}} = \mathcal{P}_{,T}(\delta T - \dot{T}\delta\tau) + \dot{\mathcal{P}}\delta\tau, \\ \delta\mathcal{E} &= \partial_\tau\mathcal{E}\delta\tau + \delta\mathcal{E}_{\text{rf}} = \mathcal{E}_{,T}(\delta T - \dot{T}\delta\tau) + \dot{\mathcal{E}}\delta\tau,\end{aligned}\tag{2.36}$$

where we have eliminated the partial derivatives w.r.t. τ using eqs (2.34). Using the definition of the sound speed in the rest frame, as we had defined it for the adiabatic fluid, we can combine the above two results to obtain,

$$\delta\mathcal{P} = c_s^2\delta\mathcal{E} + (\dot{\mathcal{P}} - c_s^2\dot{\mathcal{E}})\delta\tau.\tag{2.37}$$

This relation has a very similar form to that for the non-adiabatic fluid, eq. (2.33), which was defined as being dependent on at least two thermodynamic potentials and with no explicit dependence on time. However, its meaning is *completely* different: here the substance depends only on *one* thermodynamic potential, but the way it depends on it evolves as a function of the rest-frame time τ . Thus it is not the same fluid at two different times. Usual hydrodynamics does *not* apply in these circumstances. Or more bluntly put: the phase diagram of such substances would not be the same at different points in time.

If we now assume that at the background level the rest frame and the observer's frame are the same, the transformation between the two frames is small. The relative perturbations of coordinates are linear. Since τ is the proper time for the rest frame, we can think of the velocity of the rest frame as the derivative

$$U_\mu = -\partial_\mu\tau,\tag{2.38}$$

with the normalisation automatically taken care of. We can thus arrive at the standard expression [66] for the pressure perturbation of this substance

$$\delta\mathcal{P} = c_s^2\delta\mathcal{E} + \dot{\mathcal{E}}\left(c_a^2 - c_s^2\right)\frac{V}{k^2},\tag{2.39}$$

where we have defined the ‘‘adiabatic sound speed’’ c_a^2 as $\dot{\mathcal{P}}/\dot{\mathcal{E}}$ and $V = \partial_i\delta U^i$ is the divergence of the velocity field perturbation. We should underline once more: this expression is not hydrodynamical. It explicitly depends on the coordinates of the frame in which the observations are being performed. V should *not* be thought of as an entropy perturbation.

2.4 k-essence closure relations

We have spent quite some time laying out the hydrodynamical picture in order to facilitate a comparison with a classical scalar-field theory. Here we will discuss general *k-essence*, defined by the Lagrangian

$$\mathcal{L} = K(\phi, X).\tag{2.40}$$

In the scalar-field variables, the EMT can be written as

$$T_{\mu\nu} = (mK_m - K)\frac{\nabla_\mu\phi\nabla_\nu\phi}{m^2} + K\left(g_{\mu\nu} + \frac{\nabla_\mu\phi\nabla_\nu\phi}{m^2}\right),\tag{2.41}$$

where we are using the definition $m \equiv \sqrt{2X}$. We've written the EMT in such a way that allows us to easily see that it takes a perfect-fluid form when we identify the gradient of the scalar with the rest-frame velocity, i.e. $u_\mu = \nabla_\mu \phi / m$. As we have already stated in section 2, this choice identifies ϕ with a comoving clock for the scalar system and therefore $m = u^\mu \nabla_\mu \phi$ with the rate of flow of this time. There is no other choice of frame we could have made for this model, since only with this choice of u_μ is the EMT of perfect-fluid form. The EMTs of more general scalar-field models cannot be rewritten in perfect-fluid form at all [58].

It is important to stress that this perfect-fluid description does *not* always apply for the EMT (2.41). As we have stated in section 2.3 the velocity u_μ needs to be time-like so that it can represent the velocity of fluid elements. A configuration of a scalar field can in general have non-time-like gradients (for example, a static domain wall). There is nothing wrong with such a configuration: its evolution is described by the equation of motion for the scalar field in the usual way. However, the EMT for such can *not* be interpreted as one of a perfect fluid. Similarly, the EMT of an oscillating scalar configuration is not of perfect-fluid form at all times. The velocity $\dot{\phi}$ vanishes and reverses direction. This means that at the rate of flow of time vanishes and reverses and the scalar reaches the maximal amplitude. In between two maxima, we can interpret the scalar as a perfect fluid, but this interpretation breaks down at the maxima themselves. Of course, it is well known that an oscillating scalar field (for example, at the end of inflation) effectively evolves as pressureless dust. Indeed this is true. However, there we are making a statement about the properties of the EMT averaged over oscillation cycles. This requires the existence of an external clock which describes the oscillatory solutions and therefore the existence of an external reference system which increases monotonically. The EMT of the scalar in such a situation is not that of a perfect fluid at all times, it is just that the particular solution can be described by an averaged EMT which has the form of the EMT for dust.

Perfect-fluid evolution can always be written in terms of a non-canonical scalar-field theory [79]; however, not all solutions of *k-essence* theories can be described by perfect fluids. The reason why fluid descriptions are at all useful in cosmology is that the symmetry of the FLRW background requires that the spatial gradients vanish. Thus particular solutions always have time-like velocities (apart from the oscillatory cases mentioned). Linear perturbation on this background by definition do not change this property since they are small. However, the question of whether the particular solution behaves like a fluid is in principle very different from the question of whether the scalar-field itself is a fluid. As we show here, this is not always the case and therefore it should not be a surprise that one does not in general obtain hydrodynamical relations between the perturbations of pressure and energy density. We will show this explicitly now.

For *k-essence* models, both the pressure and the energy density are functions of just two variables, ϕ and m . As we discussed on page 13, it is natural to first write down the perturbations in terms of these underlying independent variables,

$$\begin{aligned}\delta\mathcal{P} &= \mathcal{P}_m \delta m + \mathcal{P}_\phi \delta\phi, \\ \delta\mathcal{E} &= \mathcal{E}_m \delta m + \mathcal{E}_\phi \delta\phi.\end{aligned}$$

where the subscripts m, ϕ denote partial differentiation w.r.t. those variables. We can then

eliminate the derivatives w.r.t. ϕ by taking time derivatives of \mathcal{E} and \mathcal{P} , obtaining

$$\begin{aligned}\delta\mathcal{P} &= \mathcal{P}_m \left(\delta m - \frac{\dot{m}}{m} \delta\phi \right) + \dot{\mathcal{P}} \frac{\delta\phi}{m}, \\ \delta\mathcal{E} &= \mathcal{E}_m \left(\delta m - \frac{\dot{m}}{m} \delta\phi \right) + \dot{\mathcal{E}} \frac{\delta\phi}{m}.\end{aligned}\tag{2.42}$$

Given the definition of the spatial velocity divergence Θ , eq. (2.20), we can combine the above to obtain the *k-essence* closure relation

$$\delta\mathcal{P} = C^2 \delta\mathcal{E} + \dot{\mathcal{E}} \left(c_a^2 - C^2 \right) \frac{\Theta}{k^2},\tag{2.43}$$

where $C^2 \equiv \mathcal{P}_m/\mathcal{E}_m$. This relationship is *not* the hydrodynamical closure relation for a non-adiabatic fluid (2.33), but rather it is of the type for a substance depending explicitly on its rest-frame time (2.39), exactly since it depends on the perturbation of the rest-frame represented by Θ . Even assuming that the gradients $\nabla_\mu \phi$ are time-like, a general *k-essence* theory is not hydrodynamical, but represents a fluid which itself changes with time. There is only one thermodynamical potential, m , and there are no entropy perturbations in this model. However, in the shift-symmetric case, $K = K(X)$, there is no explicit ϕ dependence, as a result of which $c_a^2 = C^2$. In this case, the coordinate dependence of the closure relation (2.43) disappears and we recover the standard behaviour for a perfect adiabatic fluid, eq. (2.31). However, this is a fluid which can only carry scalar perturbations and not vorticity: a superfluid.⁸

One final aspect that needs to be checked is the relation of C^2 to what we call the physical sound speed c_s^2 . C^2 relates the pressure and energy-density perturbation through eq. (2.43), and therefore determines the Jeans length. What we will call the *physical* sound speed c_s^2 in the case of a scalar-field theory is defined as the speed of propagation of small perturbations of the *field*. This physical sound speed determines the rate at which information propagates and therefore the causal structure of the scalar medium [80]. It is obtained by calculating the effective metric for perturbations. We show how this can be done in section 3.2. In general the two quantities do not have to be equal, but, in the case of *k-essence* $C^2 = c_s^2$ for both shift-symmetric and general models. The Jeans length in *k-essence* is therefore always determined by this physical sound speed c_s^2 , rather than any other speed of pressure waves [81]. This also could have been a different result and indeed it is highly non-trivial to obtain it in more complicated theories. However, because of the link between the Jeans term and causality, resulting from the fact that it describes the reaction of the fluid to changes in the configuration, it is this physical sound speed of field perturbations that will in the end determine the behaviour of the scalar fluid at small enough scales.

In conclusion: *k-essence* models hydrodynamics of an adiabatic fluid only in the shift-symmetric case. Away from shift symmetry, the substance it models changes its properties as a function of time. However, it still depends only on one thermodynamical potential and does not carry entropy perturbations. One can see that a non-adiabatic perfect fluid would be modelled by two shift-symmetric scalar fields. The two magnitudes of the gradients would represent two different thermodynamic potentials, while the independence from the

⁸As we have already stated, this is a result of the Frobenius theorem. The rotation tensor for the velocity vector field u^μ vanishes, $\omega_{\mu\nu} \equiv \nabla_{[\mu} u_{\nu]} = 0$, since u_μ is a derivative of a scalar. The vorticity vector for the vector field u_μ is then defined as $\omega^\mu \equiv \varepsilon^{\alpha\beta\gamma\mu} \omega_{\alpha\beta} u_\gamma$ and therefore also is zero.

field values would ensure that the fluid were unchanging. The description of perfect-fluid hydrodynamics through scalar theories was discussed in ref. [79].

As we will start to show in the next section, the closure relations for more complicated models depart even further from the hydrodynamical picture. Nonetheless, just as in the case of eq. (2.42), hydrodynamic quantities such as the equation of state and the sound speed do appear in the closure relations and therefore one can hope to express the general behaviour using a small set of physical parameters. We shall therefore be very careful when deriving the closure relations and will show explicitly that the correct relations are *not* those of hydrodynamics.

Let us invest a little time to rewrite the perturbations for *k-essence* (2.42) using a different set of variables, which will make the transition to more difficult cases discussed in section 3 more transparent. The pressure perturbation can be considered to be a sum of two parts: spatial and temporal,

$$\frac{k^2}{a^2}\delta\mathcal{P} = m\mathcal{P}_m A + \dot{\mathcal{P}}\Theta. \quad (2.44)$$

with a similar result for the energy-density perturbations. We have defined the variable A ,

$$A \equiv \left(\frac{k^2}{a^2} \frac{\delta m}{m} - \frac{\dot{m}}{m} \Theta \right) = \dot{\Theta} + 2H\Theta - \frac{k^2}{a^2} \Psi, \quad (2.45)$$

to represent the perturbation of m as it would be seen in the scalar frame u^μ , i.e. it is the equivalent of the rest-frame perturbation (2.35). It is also equivalent to the spatial divergence of the perturbation of the acceleration a_λ (see eq. (2.7)). Spatial in this case implies independent of the perturbation of the clock ϕ .

We then obtain the closure relations by rewriting the velocity divergence Θ in eq. (2.44) through the total energy flux Ξ (eq. (2.21)) and then eliminating A from the pressure perturbation equation using the energy-density perturbation. We have to do this since only Ξ and not A has an evolution equation (eq. (2.23)) in the language that we have set up. In this language the closure relations (2.43) for *k-essence* become

$$\text{k-essence: } \delta\mathcal{P} = c_s^2 \delta\mathcal{E} + 3(c_s^2 - c_a^2) \left(\frac{aH}{k} \right)^2 \frac{\Xi}{H}, \quad (2.46)$$

$$\frac{k^2}{a^2} \delta\pi = 0. \quad (2.47)$$

The horizon suppression factor, aH/k , appears here since Ξ was defined in eq. (2.21) as containing two spatial derivatives already and this factor cancels out these derivatives. The *k-essence* model contains just functions of ϕ and m and does not contain any higher-order derivative terms, therefore the perturbations themselves cannot contain such contributions. This changes in theories containing second derivatives of the scalar in the Lagrangian. Apart from this, the coefficients are just functions of time, dependent purely on the evolution of background quantities. It is only such terms that can reasonably be hoped to be constant or to scale simply when the scalar is following a scaling solution. Our aim in the next section is to extract this sort of behaviour for more general models.

2.5 General parameterisation of closure relations

As we have already mentioned, in order to properly solve for the evolution of energy-density perturbations in any particular model in the fluid language, we need to calculate the appropriate closure relations for it. At this stage it is useful to introduce a parameterisation

for these relations. This parameterisation will be based on the expectations of the kind of terms that more general theories of a single scalar degree of freedom yield. Our aim is not to introduce a set of parameters that will span the space of all possible and impossible theories and which should be constrained from the data. After all, the most general theory of a single scalar degree of freedom is already known [49, 50]. What remains is to perform calculations for these healthy theories within some a physically meaningful framework. Our parameterisation should be thought of as a book-keeping device which should aid in these calculations. As we will show, the advantage of this approach, as opposed to just solving the equations of motion for the scalar fields, is that it allows for understanding the effect of changing the hydrodynamical properties such as the sound speed and equation of state, within particular classes of models, without worrying about the exact form of the Lagrangian.

We propose that, under the assumptions we detail below, the form of the closure relations will be contained within the structure:

$$\begin{aligned}\delta\mathcal{P} &= C^2\delta\mathcal{E} + 3\Sigma_1\left(\frac{aH}{k}\right)^2\frac{\Xi}{H} + \Sigma_2\frac{\Xi}{H} + \beta_{\mathcal{P}}e^{-\varkappa}\delta\rho + \gamma_{\mathcal{P}}\frac{e^{-\varkappa}\rho}{H}\Theta_{\text{m}}\left(\frac{aH}{k}\right)^2, \\ \frac{k^2}{a^2}\delta\pi &= \Pi\delta\mathcal{E} + 3\varpi_1\left(\frac{aH}{k}\right)^2\frac{\Xi}{H} + \varpi_2\frac{\Xi}{H} + \beta_{\pi}e^{-\varkappa}\delta\rho + \gamma_{\pi}\frac{e^{-\varkappa}\rho}{H}\Theta_{\text{m}}\left(\frac{aH}{k}\right)^2.\end{aligned}\quad (2.48)$$

where ρ and $\delta\rho$ are the energy density and its perturbation of any matter external to the dark energy assumed here to behave as dust, Θ_{m} is the velocity divergence of the external matter, e^{\varkappa} is the effective Planck mass which appears as a result of our having eliminated $G_{\mu\nu}$ from the EMT of the dark energy. The set of parameters which need to be *calculated* for a particular class of models are: C^2 , Π , $\Sigma_{1,2}$, $\varpi_{1,2}$, $\beta_{\mathcal{P},\pi}$ and $\gamma_{\mathcal{P},\pi}$.

Let us justify this structure. Firstly, it is clearly not a structure which would be given by hydrodynamics of a perfect fluid, just as it wasn't in the case of *k-essence*. In general, the EMT will contain standard *k-essence* contributions, which will result in the type of terms obtained in eq. (2.46). More general scalar models contain second derivatives in the EMT, generating entries in all the components, as can be seen from the decomposition (2.11). The fluid variables can for example depend on the expansion, θ or the intrinsic curvature of the spatial hypersurface, ${}^{(3)}R$. When perturbed these give

$$\begin{aligned}\delta\theta &= \Theta + 3(\dot{\Phi} - H\Psi), \\ \delta{}^{(3)}R &= 4\frac{k^2\Phi}{a^2} - 4H\Theta.\end{aligned}\quad (2.49)$$

The gravitational potentials here can be solved for by using the Einstein equations (2.24), at the cost of introducing a dependence on the energy density and velocity divergence of any external matter present in the cosmology. Thus in general models with second derivatives, the perturbed fluid variables will not only be a function of the scalar configuration variables $\delta\phi$ and δm , but also be functions of the configuration of the external matter. This is the origin of the maybe surprising parameters $\beta_{\mathcal{P},\pi}$ and $\gamma_{\mathcal{P},\pi}$ in the parameterisation (2.48). This doesn't make the system unsolvable, since there will also be conservation equations for the external matter equivalent to eqs (2.22) and (2.23), but it makes these more-general models significantly more complex. eqs (2.49) show that the dependence on Θ arising from the second-derivative terms is unsuppressed by k/aH , leading to the contributions of terms Σ_2 and ϖ_2 to the parameterisation (2.48).

Another general feature that appears in these models with second derivatives is the presence of $dm/d\tau$, the second time derivative of the scalar field, in quantities such as \mathcal{P} (see

e.g. eq. (3.8) or ref. [58]). This is yet again another independent variable on which the pressure perturbation will depend. The method we propose for solving this complication is to use the equation of motion of the scalar field to eliminate $dm/d\tau$, prior to perturbing. Since any coupling to external matter will by definition also appear in this equation of motion, this sort of replacement will again give a contribution to the pressure of dark energy EMT from the external matter. In the equation of motion this coupling can be more general than just to the energy density of the external matter.⁹ We will neglect the pressure of the external matter in this paper and thus have not included it in eqs. (2.48).

In section 3 we illustrate the above somewhat abstract arguments with a concrete example model. Even this simplest example will require most of the structure of the parameterisation (2.48). We will see that theories with second-derivatives in the EMT feature a new scale for perturbations: it is determined by the relative sizes of the *k-essence* and second-derivative terms in the perturbed EMT. At large enough scales, the *k-essence* terms will dominate and the perturbations will evolve as they do in *k-essence*. At smaller scales, the two-derivative terms will dominate bringing new features to the evolution of the perturbations, such as anisotropic stress. This means that the parameters introduced in (2.48) are not strictly speaking functions of just the background. They will contain exactly this perfect/imperfect transition scale, and will in general interpolate between two values either side of this scale. However, the way that we have defined them, away from this transition scale they will behave as constants in k (not necessarily in time). In particular, no higher derivative terms are possible (higher powers of k/aH) if we limit ourselves to models with no more than two derivatives. We should also note that each class of Lagrangians with second derivatives will introduce its own transitions scale, dependent on the value of the coefficient it carries. In this paper, we shall only investigate a toy model with one such term.

We should remark that this transition between the two limits is in the end determined by the relative importance of the Θ and δq terms in Ξ , eq. (2.21): the dominance of the δq terms signifies the dominance of the two-derivative terms. It will clearly occur at small-enough scales, when the mode k is high. However, as the form of Ξ , eq. (2.21), shows, the q terms will also dominate whenever the equation of state is close enough to that of the vacuum, $\mathcal{E} + \mathcal{P} = 0$. Thus in models which would fit the observed expansion history, having $w \simeq -1$, the transition to the imperfect behaviour will lie at relatively large scales even when the two-derivative corrections are relatively suppressed and do not contribute significantly to the evolution of the background. It is interesting to note that this transition scale is not present in the equation of motion for the scalar. It is purely a result of the structure of the EMT. However, since it is the EMT that determines the behaviour of the gravitational potentials, this scale is capable of having an observable effect.

We can now insert the parameterisation (2.48) into the perturbation equations (2.22)

⁹e.g. in Brans-Dicke theories, the equation of motion couples the scalar field to the trace of the external EMT, i.e. $\rho - 3p$, but it is a different combination of pressure and energy for galileon terms.

and (2.23) to obtain

$$\delta\mathcal{E} + 3H(1 + C^2)\delta\mathcal{E} + \left(1 + 3\Sigma_2 + 9\Sigma_1 \left(\frac{aH}{k}\right)^2\right)\Xi = \quad (2.50)$$

$$\begin{aligned} &= S_1 - 3H\beta_{\mathcal{P}}e^{-\varkappa}\delta\rho - 3\gamma_{\mathcal{P}}e^{-\varkappa}\rho\Theta_{\text{m}} \left(\frac{aH}{k}\right)^2 \\ &\dot{\Xi} + \left(5 - 3\left(\Sigma_1 + \frac{2}{3}\varpi_1\right) - \frac{\mathcal{E} + \mathcal{P}}{H^2}\varpi_2 - \left(\frac{k}{aH}\right)^2\left(\Sigma_2 + \frac{2}{3}\varpi_2\right)\right)H\Xi - \\ &- (\mathcal{E} + \mathcal{P})\left(\Pi - \frac{1}{2}\right)\delta\mathcal{E} - \left(C^2 + \frac{2}{3}\Pi\right)\frac{k^2}{a^2}\delta\mathcal{E} = \\ &= S_2 + \left(\frac{k^2}{a^2}\left(\beta_{\mathcal{P}} + \frac{2}{3}\beta_{\pi}\right) - \frac{(\mathcal{E} + \mathcal{P})}{2}\right)e^{-\varkappa}\delta\rho + H(\gamma_{\mathcal{P}} + \frac{2}{3}\gamma_{\pi})e^{-\varkappa}\rho\Theta_{\text{m}} \end{aligned} \quad (2.51)$$

where we have explicitly included the non-conservation terms $S_{1,2}$ which appear as a result of the evolving Planck mass in the presence of external matter, eq. (2.13),

$$\begin{aligned} S_1 &= \delta\left(e^{-\varkappa}T_{\mu\nu}^{\text{ext}}u^{\mu}\nabla^{\nu}\varkappa\right), \\ S_2 &= -ik^i\delta\left(e^{-\varkappa}T_{\mu\nu}^{\text{ext}}\nabla^{\mu}\varkappa\perp_{\lambda}^{\nu}\delta_i^{\lambda}\right) \end{aligned} \quad (2.52)$$

where the δ signifies that the term inside the parentheses must be perturbed. We have also dropped some terms involving the gravitational potential, subdominant subhorizon (i.e. negligible on scales $k \gg aH$, which are the most interesting ones from the observational point of view). Given this form, we can combine the two equations to eliminate the variable Ξ and obtain a single second-order differential equation for $\delta\mathcal{E}$ with sources related to the presence of external matter. Let us make some simplifications first:

- The term in eq. (2.51) containing $\Sigma_2 + \frac{2}{3}\varpi_2$ will contribute to the friction term in the evolution equation for $\delta\mathcal{E}$. When rewritten in $\ln a$ as the time coordinate, it can be easily seen that this will be a contribution exponentially growing in time, large at small scales. Depending on the sign, it either is an anti-friction term leading to small-scale instability or a very quickly growing friction term, destroying any perturbation growth. In the model we describe in (3), it is never important and vanishes on small scales. We suspect that this is in general the case for all consistent theories. We will set it to zero in what follows.
- Since the pressure and anisotropy perturbations contain the external matter perturbation, $\delta\rho$, the Jeans term in eq. (2.23) gives a scale-dependent source for the DE perturbation. We can estimate from eq. (2.51) that, at small enough scales, the *particular solution* of the differential equation for $\delta\mathcal{E}$ we will approximately have is

$$\delta\mathcal{E} \simeq \frac{\beta_{\mathcal{P}} + \frac{2}{3}\beta_{\pi}}{C^2 + \frac{2}{3}\Pi}\delta\rho \quad \text{when } \beta_{\mathcal{P}} + \frac{2}{3}\beta_{\pi}, C^2 + \frac{2}{3}\Pi \gg \left(\frac{aH}{k}\right)^2. \quad (2.53)$$

These will be the solution *provided* that the homogeneous modes decay quickly enough. This is usually referred to as the *quasi-static* limit. Since in this configuration the dark-energy energy density tracks the dark matter one, this would be interpreted as a modification of the effective Planck constant in the Poisson equation, or a fifth force, and

is the sort of solution taken as the $f(R)$ modified-gravity regime. The rather interesting observation is that in models with a DE/DM coupling, it is on scales *smaller* than the Jeans length that we observe such effective modifications of the Newton's constant. It is *below* the Jeans length that the dark energy is clustered in models where it is coupled to the DM. However, we must be careful as to whether this quasi-static solution neglecting the solutions to the homogeneous equations is in fact dominant. This may not always be the case.¹⁰

Neglecting the external energy density for the moment, we can obtain an evolution equation for the energy-density perturbation of the dark energy when it is isolated:

$$\begin{aligned} \ddot{\delta\mathcal{E}} + H \left(8 + 3 \left(C^2 - \Sigma_1 - \frac{2}{3}\varpi_1 \right) + \frac{2\dot{H}}{H^2}\varpi_2 \right) \dot{\delta\mathcal{E}} + \\ + (1 + 3\Sigma_2) \left(C^2 + \frac{2}{3}\Pi \right) \frac{k^2}{a^2} \delta\mathcal{E} + H^2 \left[\frac{\dot{H}}{H^2} (4(1 - \Pi) - 3\Sigma_2 (1 - 6\Pi)) + \right. \\ \left. + (3 - 2\Pi) \left(5 + \frac{2\dot{H}}{H^2}\varpi_2 - 3 \left(\Sigma_1 + \frac{2}{3}\varpi_1 \right) \right) \right] \delta\mathcal{E} = 0 \end{aligned} \quad (2.54)$$

where we have dropped terms suppressed below the horizon and assumed Σ_2 and C^2 vary slowly.

At this stage let us stress that it is the combination of parameters $(1 + 3\Sigma_2)(C^2 + 2/3\Pi)$ that determines the Jeans length. As we will see, in our example model, it is this *combination* that will reduce to the physical sound speed squared, c_s^2 at small enough scales. Thus both the Jeans length and the speed of propagation of perturbations of the scalar will be determined by the same physical sound speed rather than a simple parameter relating $\delta\mathcal{P}$ and $\delta\mathcal{E}$.

We would also like to draw the reader's attention to the coefficient of the friction term in eq. (2.54). It contains corrections $\varpi_{1,2}$ which appear whenever the anisotropic stress is present. When the fluid is imperfect, the friction coefficient is changed (in our example it is reduced) and this modifies the power law with which the homogeneous modes of eq. (2.54) evolve, making $\delta\mathcal{E}$ decay more slowly (and the density contrast grow significantly faster) compared to its behaviour in a perfect regime. In hydrodynamics, viscous stresses act to increase the friction, so this result is very much against that intuition, underlining the non-hydrodynamical behaviour of these scalar models.

We will now turn to a discussion of a particular model that features second derivatives in the EMT within the framework proposed in this general discussion.

3 Worked example: non-minimally coupled k-essence

The (re-)discovery of galileon-type theories [43–50] has significantly extended the model-building toolbox to include theories which describe imperfect fluids and can stably violate the null energy condition [58], allowing for the first time the study of models which have such phenomenology in a consistent manner. The fundamental difference as to more phenomenological approaches, is that these theories do have a well-defined Lagrangian and equations

¹⁰This is, at least in spirit, similar to the “adiabatic instability” in coupled models [82–85]. This instability is a result of assuming that the scalar field follows the minimum of an effective potential adiabatically, i.e. its evolution is quasi-static and is determined by the density perturbation of dark matter. However, there can exist circumstances when that is not the solution which the scalar obeys which is signified by a negative effective sound speed squared of the combined DM/DE fluid.

of motion which allow for a precise understanding of the evolution. Indeed cosmological perturbation theory in these models has already been extensively studied [86–97]. However, in all these models, the presence of second-order derivatives in the EMT complicates matters very significantly, obscuring the physical meaning of the results.

The purpose of this paper is to show how one can use intuitive hydrodynamical language to study these more complicated theories. However, in the interest of simplicity and familiarity, we will study a much simpler model than the full Horndeski-type Lagrangian [50]: a *k-essence* theory non-minimally coupled to gravity. Again, this class of models has been studied before [98, 99], however from a very different point of view. We will focus on the fact that in such theories second derivatives in the EMT are generated through the non-minimal coupling to gravity. As a result of this, in the Jordan frame, this scalar theory can be seen to exhibit many of the properties of general galileon EMTs at the level of linear perturbations on a FLRW background. As we will see, this theory also features a rest frame different to the scalar frame as well as non-zero viscous stress thus addressing most of the possible structures in the more general theories. This worked example will allow us to demonstrate explicitly how one goes about deriving the parameters of the closure relations (2.48), resulting in a description which depends on physical properties of the fluid rather than a particular choice of Lagrangian to which the equation of motion is sensitive.

As we have already mentioned in section 2.2, these models are usually described in the Einstein frame, where the coupling to gravity is removed at the price of a coupling between external matter and the scalar. In the Einstein frame, the second-derivative terms in the EMT are not present and the DE fluid is just a standard *k-essence* coupled to external matter. We choose to remain in the Jordan frame, since we are using this worked example to illustrate the features of more-complex models, such as kinetic gravity braiding, where no frame redefinition exists which would remove the second-derivative terms.

3.1 General properties of energy-momentum tensor

Our worked example is described by the following action:

$$S = \int d^4x \sqrt{-g} \left[\frac{1}{2} M_{\text{Pl}}^2 e^{\varkappa(\phi)} R + K(\phi, X) + \mathcal{L}_{\text{ext}} \right]. \quad (3.1)$$

The first term defines the gravitational part of the action, which is the usual Einstein-Hilbert term but non-minimally coupled to a scalar ϕ through the function $\varkappa(\phi)$. The second term, $K(\phi, X)$, is the usual *k-essence* term: a non-canonical kinetic term for the scalar field. Further, through \mathcal{L}_{ext} we have indicated the possibility of some matter content external to the scalar field, which we do not consider in detail in this paper. M_{Pl} denotes the fundamental Planck mass, implying that the effective Newton’s constant is

$$G_{\text{eff}} = G_{\text{N}} e^{-\varkappa(\phi)}. \quad (3.2)$$

We will use units in which $M_{\text{Pl}}^2 = (8\pi G_{\text{N}})^{-1} = 1$. The always-positive form of the non-minimal coupling, e^{\varkappa} , ensures that the effective Newton’s constant does not change sign. If we start the system off in a configuration where the gravitational sector does not have a ghost, it will never evolve to destabilise it.

We will simplify the notation somewhat by using the overdot to signify the differentiation w.r.t. the scalar-frame time, $\dot{f} \equiv df/d\tau$ in section 3.1 and 3.2. Once we go back to the discussion of perturbations in section 3.3, the overdot will again mean the differentiation w.r.t. background coordinate time in FLRW.

The Einstein equations are

$$G_{\mu\nu} = T_{\mu\nu}^X + e^{-\mathcal{Z}} T_{\mu\nu}^{\text{ext}}, \quad (3.3)$$

$$T_{\mu\nu}^X \equiv e^{-\mathcal{Z}} K_{,X} \nabla_\mu \phi \nabla_\nu \phi + e^{-\mathcal{Z}} K g_{\mu\nu} + e^{-\mathcal{Z}} (\nabla_\mu \nabla_\nu - g_{\mu\nu} \square) e^{\mathcal{Z}(\phi)}. \quad (3.4)$$

where $T_{\mu\nu}^X$ is what we will refer to as the EMT of the dark energy and which includes the fact that we have divided the equations through by the effective Planck mass (see the discussion on page 7). $T_{\mu\nu}^{\text{ext}}$ is EMT for any external matter. If we consider it, we will take it to be a perfect fluid,

$$T_{\mu\nu}^{\text{ext}} = (\rho_{\text{ext}} + p_{\text{ext}}) v_\mu v_\nu + p_{\text{ext}} g_{\mu\nu}, \quad (3.5)$$

and we will assume that it be conserved, $\nabla^\mu T_{\mu\nu}^{\text{ext}} = 0$. As was alluded to in the introduction to section 3, the non-minimal coupling to the Ricci scalar in the action (3.1) results in second derivatives appearing in the energy momentum tensor eq. (3.4). In this respect this model shares many features with the more complex theories featuring galileon terms or kinetic gravity braiding from the perspective of cosmological perturbation theory. The price of the varying Planck mass is the non-conservation of our choice of the DE EMT in the presence of external matter,

$$\nabla^\mu T_{\mu\nu}^X = -\dot{\mathcal{Z}} e^{-\mathcal{Z}} T_{\mu\nu}^{\text{ext}} u^\mu, \quad (3.6)$$

where the velocity u^μ is that defined in eq. (2.2). and

$$\dot{\mathcal{Z}} = m \mathcal{K}_\phi, \quad (3.7)$$

where the subscript ϕ represents differentiation with respect to that variable. We can now decompose the dark-energy EMT in the scalar frame (2.2) according to (2.14):

$$\begin{aligned} \mathcal{E} &= e^{-\mathcal{Z}} E - \dot{\mathcal{Z}} \theta, & E &\equiv m K_m - K \\ \mathcal{P} &= e^{-\mathcal{Z}} P + m^2 (\mathcal{K}_{\phi\phi} + \mathcal{K}_\phi^2) + \dot{\mathcal{Z}} \left(\frac{\dot{m}}{m} + \frac{2}{3} \theta \right), & P &\equiv K \\ q_\lambda &= \dot{\mathcal{Z}} a_\lambda = \overline{\nabla}_\lambda q, & q &\equiv -\dot{\mathcal{Z}} \\ \tau_{\mu\nu} &= -\dot{\mathcal{Z}} \sigma_{\mu\nu} = \dot{\pi} \sigma_{\mu\nu}, & \pi &\equiv -\mathcal{K} \end{aligned} \quad (3.8)$$

The two variables E and P are the energy density and pressure arising directly from the *k-essence* term in the action, $K(\phi, X)$, with the subscript m signifying partial differentiation w.r.t. m . In the spirit of eqs. (2.15) we have explicitly provided the scalar potentials for the energy flow and viscous stress, q and π .

The decomposition (3.8) explicitly shows up the imperfect corrections to the EMT, all carrying two derivatives of the scalar ϕ . In particular, we have a contribution to the energy density proportional to the expansion, a term in the pressure proportional to \dot{m} and a non-vanishing energy flow in the scalar frame, proportional to the acceleration. These are the same modifications as in the case of kinetic gravity braiding [58], which provides the motivation for us to study this otherwise well-known model. In addition to the above corrections, in non-minimally coupled scalar models, non-vanishing viscous stress appears, here in the form of shear viscosity. There is also a bulk viscosity term in the pressure proportional to the expansion θ .¹¹ All these second-order terms appear with a common coefficient, $\dot{\mathcal{Z}}$, which

¹¹Since we are starting with a Lagrangian, the system must be conservative. It should therefore be stressed that these viscosity coefficients are not the same as one would obtain through the gradient expansion. They do imply that the evolution of the system depends on the shear and expansion, similarly to the way that shear and bulk viscosity do, but is not in fact dissipative. See the discussion in ref. [58, §3.7] where such a situation is discussed in the case of the kinetic-gravity-braiding model.

effectively provides a single additional parameter controlling such modifications. The model reduces to *k-essence* plus a constant correction to the Planck mass in the limit $\varkappa = \text{const.}$

3.2 Equation of motion and effective metric for perturbations

The equation of motion for the scalar that is obtained directly by varying the action (3.1) is

$$\nabla_\mu (K_X \nabla^\mu \phi) - K_\phi = \frac{1}{2} \varkappa_\phi e^\varkappa R. \quad (3.9)$$

where we obtain the Ricci tensor from the Einstein equations (3.3),

$$R = \mathcal{E} - 3\mathcal{P} + e^{-\varkappa} (\rho_{\text{ext}} - 3p_{\text{ext}}). \quad (3.10)$$

Combining these and rewriting the second derivatives of the scalar in terms of the kinematical decomposition (2.11) we obtain for the equation of motion,

$$\begin{aligned} \left(E_m + \frac{3}{2} m \varkappa_\phi^2 e^\varkappa \right) \dot{m} + m \left(P_m + \frac{3}{2} m \varkappa_\phi^2 e^\varkappa \right) \theta + m E_\phi = \\ = \frac{1}{2} \varkappa \left(E - 3P - 3m^2 e^\varkappa \left(\varkappa_{\phi\phi} + \varkappa_\phi^2 \right) + \rho_{\text{ext}} - 3p_{\text{ext}} \right). \end{aligned} \quad (3.11)$$

Because of the presence of the second derivatives in \mathcal{E} and \mathcal{P} and the non-minimal coupling, there is now a Planck-mass-suppressed contribution which corrects the coefficients of the second-derivative terms, \dot{m} and θ . Using this form we can also explicitly see that this scalar is coupled to the trace of the external EMT. This has an important impact on the dynamics of the scalar in the presence of external matter.

An important feature of the form of the scalar equations of motion (3.11), with all the second-derivative terms explicitly collected together, is that it allows us to calculate the sound speed by identifying the effective metric for the propagation of linear perturbations. The idea is that we linearise the equation of motion around some background configuration (including gravity) and then collect all the highest derivative terms. We will obtain an operator $\mathcal{G}_{\mu\nu}$,

$$\mathcal{G}^{\mu\nu} \nabla_\mu \nabla_\nu \delta\phi + \mathcal{O}(\nabla_\mu \delta\phi, \delta\phi, \Phi, \Psi) = 0. \quad (3.12)$$

Now given the standard eikonal ansatz, $\delta\phi = \mathcal{A}(x) \exp(i\omega \mathcal{S}(x))$, with some slowly varying amplitude $\mathcal{A}(x)$ and then taking the formal limit $\omega \rightarrow \infty$, we obtain

$$\mathcal{G}^{\mu\nu} \nabla_\mu \mathcal{S} \nabla_\nu \mathcal{S} = 0. \quad (3.13)$$

The tensor $\mathcal{G}_{\mu\nu}$ is now the effective contravariant metric for the propagation of perturbations [80, App. A]. In the case of our equation of motion (3.11) on an arbitrary background, it remains a simple diagonal metric, just as in *k-essence*,

$$\mathcal{G}_{\mu\nu} = \left(\frac{E_m}{m} + \frac{3}{2} \varkappa_\phi^2 e^\varkappa \right) u_\mu u_\nu + \left(\frac{P_m}{m} + \frac{3}{2} \varkappa_\phi^2 e^\varkappa \right) \perp_{\mu\nu}. \quad (3.14)$$

Note that this is an exact expression valid for any background metric, not just, for example, for FLRW. We do have a correction to the *k-essence* result, with the non-minimal coupling to gravity resulting in M_{Pl}^2 -suppressed terms coming from the second-derivative terms in the

DE EMT. From this result, we can immediately read off the condition that the scalar degree of freedom not be a ghost,¹²

$$D \equiv \frac{E_m}{m} + \frac{3}{2}\varkappa_\phi^2 e^\varkappa > 0, \quad (3.15)$$

which comes from the coefficient of the time-time part of the acoustic metric. In addition, the sound speed for the propagation of small scalar-field perturbations is the ratio of the diagonal space components of the metric (3.14) to the time-time part, i.e.

$$c_s^2 = \frac{P_m}{mD} + 3\beta > 0, \quad (3.16)$$

$$\beta(\phi, m) \equiv \frac{\dot{\varkappa}^2 e^\varkappa}{2m^2 D} > 0$$

where we have defined a new positive coupling β arising from the non-minimal coupling to gravity. Again, the sound speed squared needs to remain positive in order to prevent short-timescale gradient instabilities from developing, since this is the quantity that enters the dispersion relation. We can rewrite (3.16) in the better-known form,

$$c_s^2 = \frac{K_X + \frac{3}{2}\varkappa_\phi^2 e^\varkappa}{K_X + 2XK_{XX} + \frac{3}{2}\varkappa_\phi^2 e^\varkappa},$$

$$\beta = \frac{\varkappa_\phi^2 e^\varkappa}{2K_X + 4XK_{XX} + 3\varkappa_\phi^2 e^\varkappa}.$$

The sound speed is a combination of the standard expression for *k-essence* and a correction from the non-minimal coupling. The sound speed is always equal to the speed of light for a canonical kinetic term. On ghost-condensate-type attractors ($P_m = 0$) the sound speed is non-zero as a result of the modification from the coupling to gravity. Thus if the non-minimal coupling to gravity is weak ($\varkappa(\phi)$ is slowly varying) and we have approximate shift symmetry in the $K(\phi, X)$ term, we would expect the solution to approximately follow the *k-essence* attractor, with $w \simeq -1$ while the sound speed would naturally be close to 3β , at least when the dark energy is dominant.

In general, the sound speed $c_s^2 > 3\beta$. This restriction could perhaps be avoided for some functions $K(\phi, X)$ involving inverse powers of X . However, these are likely to have additional pathologies in their phase space and we shall not consider such models here and assume this hierarchy.

It should be stressed that the result (3.16) is valid for the action (3.1) and is not a general prescription for the sound speed. See e.g. refs [47, 100] for the modifications to the expressions generated by galileon-type terms in the Lagrangian. For the avoidance of confusion with other uses in the literature, we will refer to eq. (3.16) as the *physical* sound speed. The key remark here is that this metric (3.14) describes the effective space time (and its causal structure) that a perturbation in the scalar field sees. Thus signals sent using the scalar field would propagate in this metric and therefore with the speed (3.16).

One should ask about the relevance of the speed of propagation of scalar signals to what is normally considered in the study of cosmological perturbations: the sound speed is normally taken to relate the pressure and energy-density perturbations; frequently it is even

¹²This is in addition to the condition $G_{\text{eff}} > 0$, which is required to keep the gravity sector healthy.

defined as $\delta\mathcal{P}/\delta\mathcal{E}$. One speaks about the speed of pressure waves with the idea that the response from pressure is what stops collapse and gives the Jeans length. As we will show, in the model (3.1) this reaction to collapse is not just driven by the pressure \mathcal{P} alone, but also by the anisotropic stress, as was already previewed in eq. (2.54). Moreover, we will show that despite the fact that the relationship between the pressure and energy-density perturbations for scalar-field theories is in general complicated, the Jeans length is determined by the actual physical sound speed (3.16). Thus the Jeans length, even in these more complex theories is determined by causality and the rate of propagation of signals in the scalar fluid. There is only ever one physical sound speed and it is determined by the configuration of the cosmological background and therefore, in FLRW cosmology, can at most be a function of time.

As we show in section 3.3, the quantity β is associated with the existence of a new scale delimiting two behaviours of cosmological perturbations: it separates perfect and imperfect regimes. We will show that this is a separate scale from the Jeans scale discussed above and does not appear in the equation of motion. It only arises as a result of the structure of the EMT.

We shall now return to the discussion of perturbations in the fluid language. As we have stated in section 2.5, we need to eliminate any terms in the EMT which are not defined on the spatial hypersurface. In our model (3.1), this means eliminating \dot{m} from the pressure in the EMT (3.8). It is easy to rewrite the equation of motion (3.11) in a form which will be directly usable:

$$\dot{\kappa}\frac{\dot{m}}{m} = -\dot{\kappa}c_s^2\theta - \frac{\dot{\kappa}E_\phi}{mD} + \beta e^{-\kappa} \left(E - 3 \left(P + m^2 e^\kappa (\varkappa_{\phi\phi} + \varkappa_\phi^2) \right) + \rho_{\text{ext}} - 3p_{\text{ext}} \right). \quad (3.17)$$

We can now use eq. (3.17) directly in eq. (3.8) to re-express the pressure as

$$\begin{aligned} \mathcal{P} = & (1 - 3\beta) \left(e^{-\kappa} P + m^2 (\varkappa_{\phi\phi} + \varkappa_\phi^2) \right) + \beta e^{-\kappa} \left(E - \frac{2mE_\phi}{\dot{\kappa}} \right) \\ & - \left(c_s^2 - \frac{2}{3} \right) \dot{\kappa}\theta + \beta e^{-\kappa} (\rho_{\text{ext}} - 3p_{\text{ext}}). \end{aligned} \quad (3.18)$$

We now see some of the structure which we previewed in section 2.5: the coupling of the scalar to external matter appears explicitly in the pressure with the coupling parameter β . Secondly, a term proportional to the physical sound speed appears as a coefficient of the second-derivative term, θ . It is this term that will dominate the pressure at small scales and determine the Jeans scale.

Another implication of the the pressure's containing \dot{m} is the presence of singularities in the phase space: as can be explicitly seen in eq. (3.18). Whenever $D \rightarrow 0$, we will also have $\beta \rightarrow \infty$ and so a number of terms in the pressure will diverge. For a cosmological background solution, if the pressure in this limit is negative, the trajectory will evolve toward a pressure singularity: \dot{H} will diverge within a finite time, while a and H remain finite [101]. This sort of behaviour can also be seen in phase spaces of models with kinetic gravity braiding in ref. [62] and arises in the same way. We do not observe this in minimally coupled *k-essence*, despite the possibility of having configurations with a vanishing D since the pressure does not contain \dot{m} directly.

Let us close this discussion of the general properties of the models (3.1) by discussing the limit to $f(R)$ gravity, which is a subclass. The action is

$$S = \int d^4x \sqrt{-g} M_{\text{Pl}}^2 f(R). \quad (3.19)$$

The standard procedure is to start off from this action and rewrite it in the Einstein frame, which explicitly shows that an extra scalar degree of freedom is present. However, even in the Jordan frame one can perform a Legendre transformation introducing a scalar explicitly and obtaining the action

$$S = \int d^4x \sqrt{-g} M_{\text{Pl}}^2 [f_\phi(\phi)R + f(\phi) - \phi f_\phi(\phi)], \quad (3.20)$$

equivalent to (3.19) [5]. This form of the action shows that $f(R)$ gravity is equivalent to our class of models (3.1) in the limit where the *k-essence* function K is purely a potential $V(\phi)$. In the Jordan frame, the kinetic term for the scalar is *purely* generated through the coupling to gravity. In the EMT (3.8), apart from the potential energy $V(\phi)$, there are only terms arising from the second derivatives and the behaviour of the fluid is always determined solely by them. In particular, we have

$$\begin{aligned} \text{no scalar ghost} \quad D > 0 \quad &\Leftrightarrow \quad f_\phi > 0, \\ c_s^2 &= 1, \\ \beta &= 1/3. \end{aligned} \quad (3.21)$$

Thus $f(R)$ is a special case of the class of models (3.1) where the Jeans scale and the transition scale associated with β are unobservable, since c_s^2 and β are both order one. In cosmology, both in the background evolution and in the perturbations, the second-derivative terms dominate the DE EMT on all scales. One can reverse this observation and note that we approximately recover the behaviour of $f(R)$ solutions, including the properties (3.21) for any Lagrangian belonging to the whole class (3.1) whenever the background configuration is such that the kinetic energy arising from the *k-essence* term is small compared to the contribution to the energy density arising from the second-derivative terms, i.e. for background cosmology when

$$e^{-\chi} (E + P) \ll \dot{\chi} H. \quad (3.22)$$

Since H contains a contribution from external matter, given everything else be fixed, this condition is more likely to be satisfied in the past, when dark matter dominates the universe, although whether this happens is solution dependent. This type of solution was studied under the name of the “strong-coupling” regime for Brans-Dicke theories in ref. [102].

When condition (3.22) is not satisfied implying that the *k-essence* terms dominate, the phenomenology of the theories (3.1) is much richer: both the sound speed and the coupling β can take small values, placing new potentially observable scales affecting the evolution of dark-energy density perturbations inside the cosmological horizon. In general, the history of the evolution of such a scalar will go through both the $f(R)$ -like regime and this *k-essence*-dominated regime. The history of this evolution is highly model dependent. Since the purpose of this paper is to propose a prescription for studying such models, we will only illustrate our method in the somewhat unphysical case of a cosmology of an isolated dark-energy fluid, corresponding to the far future, when the dark matter has been diluted away by the expansion of the universe. This will allow us to demonstrate a number of general properties for scalar theories involving second derivatives in the EMT. However, we will return to a more detailed study of the full richness of these models in the presence of dark matter in a separate work.

3.3 Cosmological perturbations

In this section, we will study the behaviour of the non-minimally coupled *k-essence* model (3.1) at the linear level in cosmological perturbations using the prescription introduced in section 2.5. For simplicity, we will only consider the case of a universe containing *solely* the scalar field. This will be sufficient to prove a number of statements we have made so far, but we will not explore the full richness of the models in this work. However, at appropriate moments, we will restore the dependence on the external matter to illustrate generic properties. We will not address the issues of non-linear perturbations and the existence of the chameleon mechanism, which can change the behaviour of the scalar at small-enough scales.

We will explicitly show that this model features a new length scale which divides the regimes of perfect and imperfect behaviour of the system. In the limit of $f(R)$ theories, this length scale, as well as the Jeans scale, lie beyond observable scales. It arises in our fluid language as a result of the dominance in the pressure and energy-density *perturbations* of terms containing second derivatives of the scalar field. Such terms are generic in other models containing second derivatives, such as the galileon or kinetic-gravity braiding, therefore we expect such a length scale and transitions between two regimes with different perturbation behaviour to also be generic. Secondly, we will prove that the Jeans scale is purely dependent on the speed of propagation of the scalar-field perturbations, the physical sound speed (3.16) and not on the naive relationship between the pressure and energy-density perturbations.

From here on now the overdot will signify the derivative w.r.t. the FLRW background coordinate time, d/dt . The prime represents the derivative w.r.t. $\ln a$.

We will now explicitly demonstrate how the closure relations in our example model reduce to the form proposed in the parameterisation (2.48). Let us start by reminding ourselves of the expressions for the DE energy density and pressure, eqs. (3.8):

$$\mathcal{E} = e^{-\varkappa} E - \frac{d\varkappa}{d\tau} \theta, \quad (3.23)$$

$$\mathcal{P} = e^{-\varkappa} \left((1 - 3\beta) \left(P + m^2 (\varkappa_{\phi\phi} + \varkappa_{\phi}^2) \right) + \beta \left(E - \frac{2E_{\phi}}{\varkappa_{\phi}} \right) \right) - \left(c_s^2 - \frac{2}{3} \right) \theta \frac{d\varkappa}{d\tau}. \quad (3.24)$$

These are *exact* covariant expressions in the absence of external matter and where the dependence on $dm/d\tau$ has been eliminated through the equation of motion (3.17). In order to be explicit, let us define the equation-of-state parameter and the adiabatic sound speed,

$$w \equiv \frac{\mathcal{P}}{\mathcal{E}}, \quad (3.25)$$

$$c_a^2 \equiv \frac{\dot{\mathcal{P}}}{\dot{\mathcal{E}}} = w - \frac{w'}{3(1+w)},$$

where the prime $'$ denotes a differentiation with respect to the logarithm of the scale factor, $\ln a$ and all the quantities above are background quantities.¹³ The w of eq. (3.25) is the equation of state for the DE; however, in the DM-less scenario being discussed here it is also the total w of the cosmological expansion.

¹³Note that the relationship between c_a^2 and w is strictly speaking only valid when no external matter is present: in the presence of external matter, the DE energy momentum tensor defined here is not conserved (see eq. (3.6)) and therefore $\dot{\mathcal{E}}$ has a new contribution dependent on ρ_{ext} . This modifies the expression for c_a^2 .

We will focus our study on configurations where

$$\begin{aligned} |\varkappa'| &\ll 1, \\ \beta &\ll |\varkappa'|. \end{aligned} \quad (3.26)$$

The first condition implies that the Planck mass be evolving slowly. The second implies that the external matter, which is universally coupled in this model, be coupled relatively weakly, so as to avoid strong violations of the equivalence principle. It also means that we are not in the well-studied $f(R)$ -like regime discussed on page 27. We are refraining from discussing issues such as Solar-System tests in this paper, only concentrating on the formalism, but the requirements (3.26) should make it more likely that the parameter space being discussed is not already ruled out [103].

We are going to treat the expressions (3.23) and (3.24) as functions of three independent variables: ϕ , m and the expansion θ . In general one would also have to include the perturbations of the internal degrees of freedom of the external matter. The Einstein equations provide some constraints between these variables. In order to obtain the perturbations for the hydrodynamic variables we are going to vary with respect to all these quantities. Eliminating the derivative w.r.t. ϕ in favour of the time derivative, we obtain

$$\begin{aligned} \delta\mathcal{E} &= \dot{\mathcal{E}} \frac{a^2\Theta}{k^2} - \varkappa \left(\delta\theta - 3\dot{H} \frac{a^2\Theta}{k^2} \right) + m\mathcal{E}_m A \frac{a^2}{k^2}, \\ \delta\mathcal{P} &= \dot{\mathcal{P}} \frac{a^2\Theta}{k^2} - \varkappa \left(c_s^2 - \frac{2}{3} \right) \left(\delta\theta - 3\dot{H} \frac{a^2\Theta}{k^2} \right) + m\mathcal{P}_m A \frac{a^2}{k^2}, \end{aligned} \quad (3.27)$$

where A is the divergence of the acceleration defined in eq. (2.45). It expresses the spatial perturbation of the scalar as seen in the scalar frame (2.2), orthogonal to the perturbation of the proper time of that frame represented by Θ . We can now use (2.49) to re-express the $\delta\theta$ in terms of Θ and A . Finally, we have to rewrite every Θ in terms of the momentum flux Ξ , for which the definition eq. (2.21) reduces in our model (3.1) to

$$\Xi = (\mathcal{E} + \mathcal{P})\Theta + \varkappa A. \quad (3.28)$$

We do this since we wish to obtain evolution equations purely in terms of $\delta\mathcal{E}$ and we are able to rewrite the EMT conservation equation to retain only the combination Ξ , eq. (2.22). Putting all these together, and provided $\mathcal{E} + \mathcal{P} \neq 0$, we obtain

$$\begin{aligned} \delta\mathcal{E} &= e^{-\varkappa} m^2 D \left(1 + \frac{2\beta}{\mathcal{E} + \mathcal{P}} \frac{k^2}{a^2} \right) \frac{a^2}{k^2} A - 3 \left(\frac{aH}{k} \right)^2 \frac{\Xi}{H} - \frac{\varkappa}{\mathcal{E} + \mathcal{P}} \Xi, \\ \delta\mathcal{P} &= e^{-\varkappa} m^2 D \left(1 + \frac{2\beta}{\mathcal{E} + \mathcal{P}} \frac{k^2}{a^2} \right) C^2 \frac{a^2}{k^2} A - 3c_a^2 \left(\frac{aH}{k} \right)^2 \frac{\Xi}{H} - \frac{\varkappa}{\mathcal{E} + \mathcal{P}} \left(c_s^2 - \frac{2}{3} \right) \Xi \end{aligned} \quad (3.29)$$

where in the coefficients we have neglected a number of terms subdominant at scales below the horizon. We will return to the definition of the quantity C^2 in eq. (3.34). Since we have no evolution equation for A , we have to eliminate the dependence of $\delta\mathcal{P}$ on it by combining with the equation for $\delta\mathcal{E}$, which gives us

$$\begin{aligned} \delta\mathcal{P} &= C^2 \delta\mathcal{E} - 3(c_a^2 - C^2) \left(\frac{aH}{k} \right)^2 \frac{\Xi}{H} - \frac{\varkappa}{\mathcal{E} + \mathcal{P}} \left(c_s^2 - \frac{2}{3} - C^2 \right) \Xi + \\ &+ \frac{3}{2} \varkappa e^{-\varkappa} \rho \left(c_s^2 - \frac{2}{3} - C^2 \right) \frac{a^2}{k^2} \Theta_m + \beta e^{-\varkappa} \delta\rho. \end{aligned} \quad (3.30)$$

where we have for the moment restored the dependence on external matter. As we have previously claimed in section 2.5, this is indeed a closure relation which has the structure of eqs. (2.48). All the parameters are purely functions of the background, apart from C^2 , which contains one scale as we shall see below. Nonetheless, this is a significantly more complex expression than that obtained for *k-essence* in eq. (2.46).

We can perform the same computation for the anisotropic-stress closure relation. In the model (3.1), the expression for the anisotropic stress is simple

$$\frac{k^2}{a^2}\delta\pi = -\dot{\varkappa}\Theta = \frac{m^2 D}{1+f} \frac{2\beta}{\mathcal{E}+\mathcal{P}} A - \frac{\dot{\varkappa}}{\mathcal{E}+\mathcal{P}} \Xi. \quad (3.31)$$

We can again obtain the closure relation of the correct structure by eliminating the A term from eq. (3.31)

$$\frac{k^2}{a^2}\delta\pi = \Pi\delta\mathcal{E} + 3\Pi \left(\frac{aH}{k}\right)^2 \frac{\Xi}{H} + (\Pi-1) \frac{\dot{\varkappa}}{\mathcal{E}+\mathcal{P}} \Xi, \quad (3.32)$$

with

$$\Pi \equiv \frac{2\beta}{\mathcal{E}+\mathcal{P}} \frac{k^2}{a^2} \left[1 + \frac{2\beta}{\mathcal{E}+\mathcal{P}} \frac{k^2}{a^2} \right]^{-1}, \quad (3.33)$$

Π interpolates between a small k -dependent quantity at large scales (effectively 0, anisotropic stress vanishes) and 1 at small scales, which signifies that the anisotropic stress be large.

Let us return to the final undefined variable in (3.30),

$$C^2 = \left[\frac{e^{\varkappa}(m\mathcal{P}_m + 3Hc_a^2\dot{\varkappa})}{m^2 D} + \frac{2\beta}{\mathcal{E}+\mathcal{P}} \left(c_s^2 - \frac{2}{3}\right) \frac{k^2}{a^2} \right] \left[1 + \frac{2\beta}{\mathcal{E}+\mathcal{P}} \frac{k^2}{a^2} \right]^{-1}. \quad (3.34)$$

where we have again neglected some terms irrelevant on subhorizon scales. This expression is clearly not exactly the physical sound speed of the scalar as was defined through the acoustic metric in eq. (3.16). However, given the assumptions (3.26), we can approximate the expression (3.34) as

$$C^2 \simeq \left[c_s^2 + M_C^2 + \left(c_s^2 - \frac{2}{3}\right) \frac{2\beta k^2}{a^2(\mathcal{E}+\mathcal{P})} \right] \left[1 + \frac{2\beta}{\mathcal{E}+\mathcal{P}} \frac{k^2}{a^2} \right]^{-1} + \mathcal{O}(\beta^2, \beta c_s^2), \quad (3.35)$$

$$M_C^2 \equiv -\frac{2\beta}{\varkappa'}(1-3c_a^2) + 4\beta \left(\frac{\varkappa_{\phi\phi}}{\varkappa_\phi^2} \right) + \frac{2\beta}{\varkappa'} \frac{E_\phi}{mHD} \left(\frac{mD_m}{D} - \frac{mE_{\phi m}}{E_\phi} \right). \quad (3.36)$$

We should stress that the above is an approximation only valid when the conditions (3.26) are satisfied and only on subhorizon scales.¹⁴ We can see that as the scale k is varied, the quantity C^2 interpolates between two limiting values, both related to the physical sound speed, c_s^2 and $c_s^2 - 2/3$. In addition, there is a modification resulting from the M_C^2 term which we should stress is defined here to be dimensionless, effectively in terms of the Hubble scale.

As we will see, M_C^2 determines one of the scales in the problem, which is equivalent to the Compton scale in $f(R)$ models. It represents the degree of breaking of shift symmetry

¹⁴In particular, these expressions are *not* valid in the limit of $f(R)$ gravity. One has to go back to the expression for the pressure (3.24) and recalculate assuming conditions (3.21) to be satisfied.

in the Lagrangian (3.1). If we make the ϕ dependence of $K(\phi, X)$ and \varkappa_ϕ sufficiently weak, then only the first term of eq. (3.36) will contribute and $M_C^2 \sim \mathcal{O}(\beta/\varkappa')$. We will refer to the scale created by M_C^2 in our perturbation as the Compton scale or wavelength. Interestingly, it appears from the definition (3.36) that M_C^2 can be of either sign and therefore that solutions with increasing and decreasing Planck mass are not equivalent. One of these signs will signify a fast tachyonic instability for the perturbations implying that the background solution chosen is inappropriate. We will assume that if the effect of this mass is at all relevant then the background solution must be such that the mass is positive.

In both C^2 and Π there appears the same scale-dependence which changes the coefficients in the evolution equations for the energy-density perturbations and therefore qualitatively changes their evolution. We shall discuss this scale in detail in the following sections.

3.3.1 New Transition Scale

Both the formula for C^2 , eq. (3.35), and for Π , eq. (3.33), feature the same new *transition scale*,

$$\left(\frac{k_T}{aH}\right)^2 \equiv \frac{\mathcal{E} + \mathcal{P}}{2\beta H^2} = \frac{3(1+w)}{2\beta}. \quad (3.37)$$

As we go across this scale, the coefficients interpolate between two values and the fluid transitions between two behaviours. eq. (3.33) implies that at large scales, $k \ll k_T$, the anisotropic stress in the fluid will be absent, while at scales $k \gg k_T$ we will have $\Pi \simeq 1$ and have anisotropic stress comparable to the energy density perturbation. Thus the scale k_T delineates the transition between what we will call the *perfect* and *imperfect* regimes. In the model (3.1) in the imperfect regime, the dark-energy fluid *always* carries anisotropic stress.

This transition scale is determined by the relative size of the *k-essence* terms as compared to the second-derivative terms in the EMT for linear perturbations. The *k-essence* terms contain $\delta\phi$ and δm , with coefficients related to the function $K(\phi, X)$. However, the second-derivative terms introduce terms containing $(k/a)^2\delta\phi$ and $(k/a)^2\delta m$ with a common coefficient \varkappa . The scale at which the two sets of terms are comparable defines (3.37). Equivalently, the transition between the two regimes is defined by the scale at which the Θ and δq terms are comparable in Ξ , eq. (2.21). Thus in the imperfect regime, it is the energy flow q terms that determine the evolution of the scalar-fluid perturbations. It is also worth stressing that it is the combination $E + P$ that matters here, i.e. only the kinetic energy stored in the *k-essence* terms, not the potential. This means, for example, that $f(R)$ models which do not have a kinetic-energy term in $K(\phi)$ are always in the imperfect regime.

It is interesting to note that this scale is only visible through the EMT and does not appear in the equation of motion (3.11). The background evolution of the value of the scalar field ϕ is determined by the background equation of motion, which has no spatial dependence. The evolution of the scalar's perturbations is then determined by the perturbed equation of motion, which can be written as

$$\ddot{\delta\phi} + 3H(1+\gamma)\dot{\delta\phi} + c_s^2 \frac{k^2}{a^2}\delta\phi + M^2\delta\phi = \varkappa\delta\rho, \quad (3.38)$$

where we have neglected the gravitational potentials subdominant subhorizon and have hidden the complexity of the expression in the coefficients M^2 and γ , the formulae for which we won't need. The gradient term for the scalar is clearly visible here and is determined by the physical sound speed c_s^2 . Provided a slow-enough variation of the coefficients, the solutions

to these equations are generically oscillatory. There is a change in behaviour as one crosses the Jeans scale, but this is the only transition. When external matter is present, it provides a source to eq. (3.38), which will induce an additional particular solution. This may or may not be dominant over the homogeneous solutions discussed, depending on the behaviour of all the coefficients.

However, in order to understand how the evolution of the scalar-field perturbation translates into a gravitational effect which would be felt by external matter one must consider the relation of the scalar perturbations to the perturbed EMT (3.23). And it is here that a single solution of the equation of motion provides two distinct behaviours. The *same* evolution of ϕ and $\delta\phi$ in time gives one behaviour to the *k-essence* part of the perturbed energy density

$$\delta\mathcal{E}_{\text{k-essence}} \simeq e^{-\varkappa} \left(E_\phi \delta\phi + E_m \dot{\delta\phi} \right), \quad (3.39)$$

and a *different* one to the contribution from the second derivatives,

$$\delta\mathcal{E}_{\text{2-diff}} \simeq -\varkappa_\phi \frac{k^2}{a^2} \delta\phi. \quad (3.40)$$

Thus the evolution of the energy-density perturbation, and therefore also the impact on the metric perturbations, can be completely different above and below the scale k_T , despite the same underlying solution for $\delta\phi$ and ϕ .

In the fluid language, we see this transition of behaviour exactly as a result of the change of the closure relations as we go across the scale k_T . The conservation equations for the EMT must in the end reduce to the equation of motion for the scalar, whatever the scale we are looking at. Since the energy-density perturbation is a different function of $\delta\phi$ at different scales, the closure relations must adjust the fluid equations in such a way that the underlying solution for the scalar perturbation is always the same whatever the scale. Conversely, the fluid equations must predict a different evolution for the energy-density perturbation at different scales, given that $\delta\phi$ does not change its behaviour. This is achieved by changing the closure relations and the coefficients in the evolution equation for the energy-density perturbation (2.54), for example by introducing an anisotropic stress.

Let us return to the discussion of the closure relations. Given the transition scale (3.37), we can approximate C^2 either side of it:

$$C^2 \simeq c_s^2 + M_C^2 \quad \text{for } k \ll k_T, \quad (3.41)$$

$$C^2 \simeq c_s^2 - \frac{2}{3} + M_C^2 \frac{k_T^2}{k^2} \quad \text{for } k \gg k_T. \quad (3.42)$$

where we have been agnostic as to the size of the Compton term, M_C^2 . According to our parameterisation, the Jeans scale is determined by $C^2 + 2/3\Pi$, eq. (2.54). Let us preview the implications of these approximations, leaving the detailed discussion for sections 3.3.2 and 3.3.3:

- If $c_s^2 \gg M_C^2$, the Jeans scale lies inside the Compton scale. We can neglect M_C^2 which is irrelevant to the dynamics of perturbations, given an appropriate background solution. The Jeans term is always determined by the sound speed.
- If $c_s^2 \ll M_C^2$, the Compton scale lies inside the Jeans scale. eq. (3.41) implies that in the perfect regime the perturbation will behave as *k-essence* but with an effective

sound speed equal to M_C^2 . The Compton mass term, as opposed to the sound speed, then determines the frequency of these oscillations. As we enter the imperfect regime, the Compton mass term will transition from playing the role of the Jeans term to that of a usual mass term, continuing to suppress the perturbations. There will now be a new scale determined by the relative size of c_s^2 and M_C^2 . The physical sound speed will only drive the dynamics below that scale. This is effectively the picture we have in $f(R)$.

We have now obtained both the closure relations, eqs (3.30) and (3.32) in the necessary form to use in the evolution equation for energy-density perturbations (2.54) we have set up. The forms of parameters C^2 and Π show that there are two regimes we need to consider, with the transition scale, k_T .

From the closure relations (3.30) and (3.32) we can now read off the values of the parameters defined in the general closure relations (2.48) for the case of the model (3.1). We have presented their values in the two regimes in Table 1. As can be seen, within each regime, all the parameters are indeed functions of the background quantities only, i.e. at most functions of time. Given this reduced set of parameters, the equivalent of eq. (2.54) for the model (3.1) is ¹⁵

$$\begin{aligned} \delta\ddot{\mathcal{E}} + H(8 + 3c_a^2 - 2\varpi_1)\delta\dot{\mathcal{E}} + \left(C^2 + \frac{2}{3}\Pi\right)\frac{k^2}{a^2}\delta\mathcal{E} + \\ + H^2\left(4(1 - \Pi)\frac{H'}{H} + (3 - 2\Pi)(5 + 3c_a^2)\right)\delta\mathcal{E} = 0, \end{aligned} \quad (3.43)$$

Of particular note is the presence of ϖ_1 in the friction term. This term interpolates between 0 and 1 and therefore reduces the friction term on scales where the anisotropic stress is present. As we show, this leads to slower-decaying homogeneous modes in this regime.

3.3.2 Perfect regime, $k \ll k_T$

In this regime, the closure relations (3.30) and (3.31) reduce to

$$\delta\mathcal{P} = C^2\delta\mathcal{E} + 3(C^2 - c_a^2)\frac{\Xi}{H}\left(\frac{aH}{k}\right)^2, \quad (3.44)$$

$$\begin{aligned} \frac{k^2}{a^2}\delta\pi &\simeq 0, \\ C^2 &= c_s^2 + M_C^2. \end{aligned} \quad (3.45)$$

These closure relations take the same form that generalised *k-essence* does (with the important replacement of c_s^2 with C^2), see eq. (2.46). The evolution equation for energy-density perturbations (3.43) reduces to

$$\delta\ddot{\mathcal{E}} + H(8 + 3c_a^2)\delta\dot{\mathcal{E}} + C^2\frac{k^2}{a^2}\delta\mathcal{E} + 3H^2\left[5 + 3c_a^2 - 2(1 + w)\right]\delta\mathcal{E} = 0, \quad (3.46)$$

The only correction to the standard *k-essence* behaviour is the fact that if the Compton wavelength is inside the Jeans length, $M_C^2 > c_s^2$, it will determine the behaviour of the

¹⁵We have neglected the ϖ_2 terms, which we can only do when $|\mathcal{N}| \ll |1 + w|$, which is implied by our choice (3.26). This is not the case for $f(R)$ models (see eq. (3.22)).

<i>Parameter</i>	<i>Perfect Regime</i>	<i>Imperfect Regime</i>
C^2	$c_s^2 + M_C^2$	$c_s^2 - \frac{2}{3} + (M_C k_T/k)^2$
Π	0	1
$C^2 + 2/3\Pi$	$c_s^2 + M_C^2$	$c_s^2 + (M_C k_T/k)^2$
Σ_1	$C^2 - c_a^2$	$C^2 - c_a^2$
ϖ_1	0	1
Σ_2	$-\frac{\mathcal{H}'}{(1+w)} \left(c_s^2 - C^2 - \frac{2}{3} \right)$	0
ϖ_2	$-\frac{\mathcal{H}'}{(1+w)}$	0
$\beta_{\mathcal{P}}$	β	β
β_{π}	0	0
$\gamma_{\mathcal{P}}$	$\frac{3}{2}\mathcal{H}' \left(c_s^2 - C^2 - \frac{2}{3} \right)$	0
γ_{π}	0	$-\frac{3}{2}\mathcal{H}'$

Table 1. Closure parameters for the model (3.1) as obtained by comparing the closure relations (3.30) and (3.32) with the general structure proposed in eqs (2.48). The model contains two regimes, perfect and imperfect, separated by the transition scale k_T , eq. (3.37). In the expressions for C^2 , we have assumed that the external matter constitutes a negligible part of the total energy density: i.e. dark-energy dominates. This table is in fact exact apart from this result for C^2 , where the approximation (3.35) would need to be recalculated when matter is present or conditions (3.26) do not hold. In the presence of external matter one would need to also take into account the non-conservation of the DE EMT, eqs (2.52). We have provided the results for the coefficients β_i and γ_i for completeness.

perturbations, determining the oscillation frequency of the solutions. We can rewrite the above as an equation for the evolution of the density contrast,

$$\delta'' + \left(\frac{1}{2} - \frac{9}{2}w - \frac{w'}{1+w} \right) \delta' + C^2 \frac{k^2}{a^2 H^2} \delta + M^2 \delta = 0, \quad (3.47)$$

$$M^2 = -\frac{3}{2} \left(1 + 2w - 3w^2 + \frac{2w'}{1+w} \right),$$

with $\delta \equiv \delta\mathcal{E}/\mathcal{E}$. We can now obtain the standard solutions to the homogeneous equations in the approximation that w is slowly varying. The Jeans length in the perfect regime is determined by the larger of c_s^2 and M_C^2 ,

$$\left(\frac{k_{\text{Jeans}}}{aH} \right)^2 = \frac{|M^2|}{C^2}. \quad (3.48)$$

1. *Above Jeans length*, $k \ll k_{\text{Jeans}}$. The tachyonic mass M^2 dominates the mass term allowing the DE perturbations to evolve as a power law, with modes

$$\delta_+ \propto a^{1+3w} \quad \text{and} \quad \delta_- \propto a^{-\frac{3}{2}(1-w)}. \quad (3.49)$$

We therefore have a growing mode only for $w > -1/3$. For $w > 1$, both the modes are growing. In this regime, during DE domination, the fluid will only cluster when the universe is not accelerating. The gravitational potential is constant.

2. *Below Jeans length*, $k \gg k_{\text{Jeans}}$. At scales smaller than the Jeans length $Ck/aH \gg 1$, pressure support will arrest any potential collapse, leading to oscillating solutions,

$$\delta = a^{-(1-9w)/4} \left(A_1 J_n \left(\frac{2Ck}{(1+3w)aH} \right) + A_2 J_{-n} \left(\frac{2Ck}{(1+3w)aH} \right) \right),$$

$$n \equiv \frac{1-9w}{2(1+3w)}.$$

with $A_{1,2}$ constants of integration and J_n Bessel functions of the first kind. Note that the horizon is shrinking during acceleration, so the modes will eventually leave this regime and begin to behave as if outside the Jeans length. Depending on the hierarchy between c_s^2 and M_C^2 , the oscillation frequency will be determined by the Compton term or the sound speed.

These results match those studies in refs [64, 104, 105] for the case of *k-essence*.

3.3.3 Imperfect regime, $k \gg k_T$

In this regime in the absence of matter, the closure relations (3.30) and (3.31) reduce to

$$\delta\mathcal{P} = \left(c_s^2 - \frac{2}{3} + M_C^2 \left(\frac{k_T}{k} \right)^2 \right) \delta\mathcal{E} + 3 \left(c_s^2 - \frac{2}{3} - c_a^2 + M_C^2 \left(\frac{k_T}{k} \right)^2 \right) \frac{\Xi}{H} \left(\frac{aH}{k} \right)^2, \quad (3.50)$$

$$\frac{k^2}{a^2} \delta\pi = \delta\mathcal{E} + 3 \frac{\Xi}{H} \left(\frac{aH}{k} \right)^2. \quad (3.51)$$

It is important to stress that the closure relations (3.50) are *not* hydrodynamic. Firstly, the mass terms M_C^2 appear in (3.50). If the Compton wavelength lies outside of the Jeans length, then the mass terms are always irrelevant in this regime. However, if the opposite is true, then the mass terms will dominate the dynamics on the outer edge of this regime.

Secondly, it is very striking that, at small scales, where one would naively expect to recover the sound speed, the ratio $\delta\mathcal{P}/\delta\mathcal{E}$ can be negative when the physical sound speed c_s^2 is small. If one were to take this as a definition of the speed of sound, one would interpret this sort of configuration as unstable. However, as we have been showing, the EMT of a non-minimally-coupled scalar field features anisotropic stress. This stress is very large, since in the imperfect regime it is of the same order as the energy-density perturbation, and it acts in such a way so as to precisely cancel out the term which would contribute to the would-be instability.¹⁶

This anisotropic stress from the perspective of hydrodynamics represents shear viscosity. This is a term appearing as a first-order correction in the gradient expansion and one would expect it to be small, lest it signify that the gradient expansion itself were invalid. Here, there are no terms higher-order in the gradient expansion to worry about: the elements of the EMT are at most exactly second order in gradients. These closure relations are a full rearrangement of the equation of motion for the scalar field at linear order in perturbations.

It is actually not so surprising that it is the combination of pressure and anisotropy perturbations that provides the support against collapse. Projecting out the spatial trace of the perturbed Einstein equations (the G_{ii} components, effectively), we obtain

$$\ddot{\Phi} + 6H\dot{\Phi} - 2H\dot{\Psi} - (6H^2 + 4\dot{H})\Psi = - \left(\delta\mathcal{P} + \frac{2}{3} \frac{k^2}{a^2} \delta\pi \right), \quad (3.52)$$

¹⁶The appropriate size comparison is between dimension-four quantities $\delta\mathcal{E}$ and $\frac{k^2}{a^2} \delta\pi$, as in eq. (3.51).

after substituting for a scale-dependent term in G_{ii} with the anisotropic stress. Thus the time evolution of the potentials reacts to the whole term in the parentheses, both pressure and anisotropic stress.

We should also note that the relationship between anisotropy and the energy perturbation of eq. (3.50) is the standard result for scalar-tensor theories: the nature of the anisotropic stress generated by the non-minimal coupling to gravity is such that it cancels the second-derivative contribution to the energy density leaving just the *k-essence* part of the energy-density perturbation. Effectively, in the imperfect regime, the dark-energy density perturbations in this model generate no lensing potential:

$$\frac{k^2}{a^2}(\Phi - \Psi) = \delta\mathcal{E} - \frac{k^2}{a^2}\delta\pi \simeq \delta\mathcal{E}_{\text{k-essence}} \ll \frac{k^2}{a^2}\Phi. \quad (3.53)$$

This property is an intrinsic property of this DE fluid and is always valid inside the imperfect regime, whatever the configuration. It is not just limited to quasi-static solutions. Some more general scalar-field theories have a coupling to gravity that cannot be conformally rescaled away and therefore the photons are sensitive to the energy perturbations of the scalar field (say the Lagrangians \mathcal{L}_4 and \mathcal{L}_5 in ref. [63]).

The evolution equation (3.43) in the imperfect regime reduces to

$$\ddot{\delta\mathcal{E}} + 3H(2 + c_a^2)\dot{\delta\mathcal{E}} + c_s^2 \frac{k^2}{a^2}\delta\mathcal{E} + H^2 \left[\frac{3}{2}(1+w)\frac{M_C^2}{\beta} + 5 + 3c_a^2 \right] \delta\mathcal{E} = 0, \quad (3.54)$$

We should stress that the coefficient of the Jeans term here is just the physical sound speed c_s^2 . There are two significant differences between this equation and its equivalent in the perfect regime (3.46): firstly the coefficient of the friction term is reduced from $H(8 + 3c_a^2)$ to $H(6 + 3c_a^2)$ as a result of the presence of anisotropic stress. This leads to a significant change in the evolution of the solutions of the homogeneous modes with the scale factor and can affect the relative importance of the solutions to the homogeneous equation and the particular solutions in the presence of external matter. Secondly, in the imperfect regime, the effective mass of the mode has contributions from the Jeans term, the Compton term and the other terms depending only on the background expansion, we will therefore have three regimes for perturbation evolution in this case.

We can now rewrite eq. (3.54) as an equation for the density contrast:

$$\begin{aligned} \delta'' - \frac{3}{2} \left(1 + 3w - \frac{2w'}{3(1+w)} \right) \delta' + c_s^2 \frac{k^2}{a^2 H^2} \delta + M^2 \delta &= 0, \\ M^2 &= \frac{1}{2} \left((1 + 3w)^2 + 3(1+w) \frac{M_C^2}{\beta} - \frac{2w'}{1+w} \right). \end{aligned} \quad (3.55)$$

We then have three potential classes of solutions, which depend on our comparing the relative sizes of the terms arising from

$$\begin{array}{ccc} \text{Compton} & \text{gravitational instability} & \text{Jeans} \\ M_C^2 & \leftrightarrow \frac{\beta(1+3w)^2}{1+w} & \leftrightarrow c_s^2 \frac{k^2}{k_T^2}. \end{array} \quad (3.56)$$

The gravitational instability term is the imperfect-regime equivalent of the tachyonic mass for a perfect fluid. We have kept the standard name, even though it could be considered

inappropriate here since this mass is always positive, irrespective of w . As in the perfect regime, if $c_s^2 \gg M_C^2$ then the Compton term will have no effect on the perturbations at all. On the other hand, the dominance of the Compton term over the gravitational instability or the Jeans terms implies that we are outside the Compton wavelength of the field and its mass is relevant for the dynamics, shielding the interactions. At small enough scales, the Jeans term will end up dominating eventually, since there are always scales shorter than the Compton wavelength. However whether there exists a region of scales where the gravitational-instability terms dominate is just a question of the particular model and the background configuration and not scales. We should note that it is possible for the physical sound speed to be large enough for the Jeans length to lie already inside the perfect regime. In this is the case and $M_C^2 \ll c_s^2$, the imperfect regime will only feature Jeans-term dominated scales.

We can now solve for the evolution of the DE perturbations in each of the three cases: Jeans term, Compton term and gravitational-term domination.

1. *Gravitational instability terms dominate:* $\beta(1+3w)^2/(1+w) \gg M_C^2, c_s^2 k^2/k_T^2$. The DE perturbations evolve as a power law with

$$\delta_+ \propto a^{(1+3w)/2}, \quad \delta_- \propto a^{(1+3w)}. \quad (3.57)$$

Just as in the case of the perfect regime, the homogeneous modes decay whenever the DE is causing acceleration, $w > -1/3$. However, here the mode which decays quicker does so only as fast as the “growing” mode in the case of the perfect fluid. There is now a new mode which decays even more slowly than in the case of the perfect fluid, c.f. eq. (3.49). This means that generically the Newtonian potential will be growing in this regime whenever the background is accelerating. The Poisson equation implies that the Newtonian potential.

$$\Phi = \frac{3}{2} \frac{a^2 H^2}{k^2} \delta_+ \propto a^{-(1+3w)/2}. \quad (3.58)$$

One needs to be careful, since these equations are only valid subhorizon and inside the imperfect regime, while the modes are leaving both and our universe is not yet in a state completely dominated by dark energy. However, this diametrically different behaviour could significantly affect late-time potentials already today. We will return to study these questions in detail in a separate work.

2. *Compton term dominates,* $M_C^2 \gg \beta(1+3w)^2/(1+w), c_s^2 k^2/k_T^2$. The behaviour of the modes depends on the overall sign of $(1+w)M_C^2/\beta$. Assuming that the Compton term be constant, we then have two solutions,

$$\begin{aligned} \frac{(1+w)}{\beta} M_C^2 > 0, \quad \delta &= a^{\frac{3}{4}(1+3w)} \left(A_1 \cos \frac{3M_C^2}{2\beta} \ln a + A_2 \cos \frac{3M_C^2}{2\beta} \ln a \right), \\ \frac{(1+w)}{\beta} M_C^2 < 0, \quad \delta &= A_1 a^{p_+} + A_2 a^{p_-}, \quad p_{\pm} \simeq \mp \frac{3}{2} \frac{(1+w)}{\beta} M_C^2, \end{aligned}$$

a decaying oscillator or a growing mode depending on the sign. In either case the gravitational potential grows for an accelerating equation of state. If the Compton term is substantial, the tachyonic solution is a very fast instability signifying that the

chosen background solution itself is unstable and should not be considered a realistic phenomenology.

3. *Jeans term dominates*, $c_s^2 k^2 / k_T^2 \gg M_C^2, \beta(1+3w)^2/(1+w)$. The solution in this regime is an oscillatory function, settling to a constant amplitude when $w < -1/3$. The leading-order behaviour at late times is

$$\delta = A_1 \cos\left(\frac{2c_s}{1+3w} \frac{k}{aH}\right) + A_2 \sin\left(\frac{2c_s}{1+3w} \frac{k}{aH}\right). \quad (3.59)$$

Thus contrary to the standard perfect-fluid behaviour, the density perturbation sub-Jeans-horizon does not decay away. Again, the Newtonian potential in this regime would grow while oscillating, until the mode exits the sound horizon and moves into regime (1) or (2).

We have shown that at all the scales inside the imperfect regime, the perturbations evolve in such a way that the gravitational potential being driven by them grows. This is of course a frame effect: in the Einstein frame, where gravity is minimally coupled no such increase would be observed. However, if we assume that the observable baryons are minimally coupled in the Jordan frame discussed here, they would indeed see a deepening potential in this late, dark-energy dominated era. However, in an accelerating universe, modes are exiting the horizon. Thus any mode starting off inside the Jeans length in the imperfect regime, will exit the imperfect regime eventually and would leave the sound horizon. Thus these sort of solutions should be thought of as being temporary for each mode, with shorter modes spending longer inside the imperfect regime and therefore inducing a scale-dependence in the dark-energy perturbation amplitude.

4 Discussion and conclusion

In this paper we have proposed a unified prescription for studying the subhorizon evolution of linear perturbations in theories of dark energy comprising a single scalar, including, for example, modified gravity models such as $f(R)$ and other models containing second-derivative terms in their EMTs. These EMTs are in general not of perfect-fluid form.

k-essence is the most general scalar theory that has an EMT of perfect-fluid form. The evolution of linear cosmological perturbations in this class of models is already well understood. However, any further modifications to the scalar's Lagrangian, such as galileon terms or even a simple non-minimal coupling to gravity, generate terms including second derivatives of the scalar in the EMT. The presence of the second-order derivatives makes it impossible to express the EMT in perfect-fluid form. A standard method for dealing with such models, as carried out in the literature, is to turn to the equation of motion for the scalar, usually concentrating on the limit of negligible time derivatives for the scalar field. We have argued that this is in fact on one hand too specific. For example, for *k-essence* any two background solutions with the same history of equation-of-state parameter w and sound speed c_s^2 will give exactly the same evolution of linear perturbations, irrespective of the actual Lagrangian, as eq. (3.47) shows. One may of course prefer one of the Lagrangians over another on the basis of naturalness, but this is not an observable if we only have access to the cosmological background and the effect of linear density perturbations on gravitational potentials. On the other hand: the neglecting of the time derivatives could prove dangerous, if the coefficient of the friction term in the density-contrast evolution equation turns out to be small enough

to allow for modes which grow sufficiently quickly on a particular cosmological background. We have, for example, explicitly demonstrated that in the non-minimally coupled *k-essence* model, the DE energy-density contrast during the DE-domination era evolves in such a way so as to cause the gravitational potential to grow, eq. (3.57). These modes should dominate the gravitational potential at late-enough times on at least some of the scales.

Our prescription allows for the study of such general models in terms of the intuitive fluid language instead of dealing directly with the equation of motion for the scalar field. Since the fluid variables such as pressure and energy density are just appropriately reorganised components of the EMT, they will contain all the information to which the gravitational field is sensitive.

As is well known, one can obtain the evolution equation for the energy-density perturbations from the perturbed conservation equations for the EMT. However, two closure relations need to be provided to do this: we need to relate both the pressure perturbation and the anisotropic-stress perturbation to the energy-density perturbation. The main result of this paper is the method for obtaining these closure relations, based on considering the actual degrees of freedom on a spatial hypersurface.

As we have explicitly shown in our example, this method produces closure relations that are not those of perfect-fluid hydrodynamics. One *cannot* assume that the relationship between the pressure and energy-density perturbations is hydrodynamical, $\delta\mathcal{P} = c_H^2 \delta\mathcal{E}$, with c_H a hydrodynamical sound speed for pressure waves, as is frequently done. We have explained in detail in sections 2.3 and 2.4 why even general *k-essence* models should not be thought of as a hydrodynamical fluid. The result is that the closure relations for general scalar-field theories take the somewhat complicated form (2.48) and cannot be reduced to a simple relationship.

One may argue that this hydrodynamical relation between $\delta\mathcal{P}$ and $\delta\mathcal{E}$ is the definition of the sound speed for pressure waves, c_H . However, in general this is not a physically meaningful quantity: in the example we have studied, where the anisotropic stress does not vanish, the Jeans length is not at all related to the quantity c_H , but is determined by the speed of propagation of perturbations of the scalar field, c_s , and therefore causality. As we explicitly show, c_s is obtained by considering the effective wave operator in the scalar perturbed equation of motion and is not immediately apparent in the expressions for the pressure perturbation. Even when c_s is simple and constant, the complexity of the closure relations we have derived shows that c_H could be an arbitrary function of time and space and therefore not at all a useful parameter for the description of the perturbations.

In order to aid the calculation of the closure relations, we have introduced a book-keeping parameterisation for them, suitable for a wide range of models containing a single scalar degree of freedom. This has allowed us to derive a general equation for the evolution of DE density perturbations in terms of these parameters which can be calculated for any particular class of scalar Lagrangians. For the purpose of illustration, we have shown explicitly how to calculate the coefficients of the closure relations (2.48) in a toy class of models, presenting the results in table 1. All the parameters which are non-vanishing for this class of Lagrangians are expressible in terms of physical quantities: w , c_s^2 , the coupling to external matter β and the Compton term M_C^2 , each of which is at most a function of time but never scale. Non-minimally coupled *k-essence* was chosen as the toy model since it is the simplest one to contain the majority of the features which are present in more-general scalar theories such as galileons. This prescription can be directly applied in the study of those more-complex

models.

All models containing terms with second derivatives in the EMT also contain a new scale k_T , eq. (3.37), determined by the relative contribution of the *k-essence* and second-derivative terms in the perturbations. In our toy example, as we move across this scale, the DE fluid transitions from *k-essence*-like perfect-fluid behaviour at large subhorizon scales to an imperfect fluid carrying anisotropic stress at small scales. Interestingly, the scale k_T does *not* appear in the perturbed equation of motion for the scalar itself, just in the EMT. On both sides of the transition scale, the scalar-field perturbation $\delta\phi$ evolves in the same manner, at least provided there isn't a scale dependence in any sources on the external-matter side. The changes in the pressure and anisotropy closure relations as we cross the scale k_T reflect simply that given the same evolution in time of $\delta\phi$, the density perturbation $\delta\mathcal{E}$ must evolve differently, since the *k-essence* and the second-derivative terms have a completely different functional dependence on $\delta\phi$. Since it is $\delta\mathcal{E}$ that sources the gravitational potential, this transition scale is physical. In addition to the transition scale, the dark energy will feature the known Jeans scale, controlled by the speed of propagation of small perturbations of the scalar field and a Compton scale related to the mass of the field. We have not studied superhorizon behaviour and the transition towards the horizon, but since the new phenomenology is related to the presence of second derivatives in the EMT, it will be irrelevant at large enough scales, where the standard *k-essence* behaviour is restored. Indeed sufficiently superhorizon the curvature perturbation is conserved in the standard way [59].

What our discussion shows is that it is more appropriate to think of the anisotropic stress as being related through the closure relation (2.48) to the density perturbation of the DE fluid rather than directly to the gravitational potentials. In general, the gravitational potential will also have contributions from other fluids present in the universe. The density perturbations in any particular fluid evolve according to the conservation equations for that fluid, and therefore in principle are independent dynamical variables. In our example model, in the imperfect regime, the closure relations imply that $k^2\delta\pi/a^2 \simeq \delta\mathcal{E}$, *irrespective* of the configuration of the external matter. It is this property of the fluid that makes the lensing potential independent of the perturbation in the scalar field, eq. (3.53). And it is only this relation that is in general independent of the other constituents of the universe.

The only reason why in $f(R)$ models this scale k_T is not seen, is that they are a special subclass of the non-minimally coupled *k-essence* models where the *k-essence* terms are absent. Thus the DE fluid is imperfect at all scales. The Compton scale determines whether for a particular mode the energy-density perturbations are large or small. At scales inside the Compton radius they are large and therefore the anisotropic stress is large. Outside — they are small and so is the anisotropic stress.

We should stress that every *class* of Lagrangian terms which generates second derivatives in the EMT (e.g. kinetic gravity braiding, etc.) will produce such a transition scale determined by the relative size of its coefficient in the background solution. Each class of terms will presumably cause slightly different physical effects and therefore each of these transitions should be potentially observable.

Finally, we should reiterate that in order to simplify the exposition, we have mostly neglected in this presentation the effect of external matter. This has allowed us to concentrate on understanding the properties of the DE fluid itself. As a result of the non-minimal coupling to gravity, the DE and dark matter are coupled in this model. As we have previewed in the closure relations (2.48), the coupling to dark matter causes the DM perturbation to appear in the expression for $\delta\mathcal{P}$ and therefore in the Jeans term. The DM perturbations

will in this way provide a scale-dependent source in the evolution equation for the energy-density perturbation $\delta\mathcal{E}$, dominant in the imperfect regime. This ensures that the density perturbations for DE are large in the imperfect regime, providing a significant modification to the gravitational potentials in which dark matter propagates.

The presence of an external energy density provides an additional complication: the way we defined our DE EMT means that in this case it is not conserved whenever the effective Planck mass evolves, i.e. in models with non-minimal coupling to gravity. As we show in the expressions for the conservation of the EMT (2.50) and (2.51), there are non-conservation terms which depend on the scalar and the dark-matter configuration. Therefore, we will have to provide a similar closure relation to (2.48), one for each of energy and momentum conservation equations. We will return to this also feature-rich discussion in a separate work.

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A Covariant decomposition of energy-momentum tensor

We decomposed the energy momentum tensor for dark energy according to

$$T_{\mu\nu}^X = \mathcal{E}u_\mu u_\nu + \mathcal{P} \perp_{\mu\nu} + 2q_{(\mu}u_{\nu)} + \tau_{\mu\nu}, \quad (\text{A.1})$$

in the scalar frame defined by the vorticity-free velocity field

$$u_\mu = -\frac{\nabla_\mu \phi}{m}. \quad (\text{A.2})$$

We can obtain evolution equations for the fluid by taking the divergence of the EMT. It is a vector equation, which again can be decomposed into a time and spatial part for the observer moving with velocity (A.2):

$$-u^\nu \nabla_\mu T_{\mu\nu}^X = \frac{d\mathcal{E}}{d\tau} + \theta(\mathcal{E} + \mathcal{P}) + a^\mu q_\mu + \nabla_\mu q^\mu + \sigma_{\mu\nu} \tau^{\mu\nu}, \quad (\text{A.3})$$

$$\perp_\lambda^\nu \nabla^\mu T_{\mu\nu}^X = a_\lambda (\mathcal{E} + \mathcal{P}) + \overline{\nabla}_\lambda \mathcal{P} + \frac{4}{3} \theta q_\lambda + \perp_{\lambda\mu} \frac{dq^\mu}{d\tau} + \sigma_{\lambda\mu} q^\mu + \perp_{\lambda\nu} \nabla_\mu \tau^{\mu\nu}, \quad (\text{A.4})$$

We have again assumed that the rotation tensor vanishes as a result of the definition of our velocity field. When the EMT for DE is conserved both equations (A.3) and (A.4) are identically zero. They then represent energy conservation (continuity equation) and momentum conservation (Euler equation) in the frame comoving with u^μ .

In this discussion, we are interested in analysing scalar-field theories. Since the scalar-field is isolated, both the energy flow and the viscous-stress tensor are formed from appropriate derivatives of some scalar-field potentials, namely

$$q_\lambda = \perp_\lambda^\mu \nabla_\mu q = \overline{\nabla}_\lambda q, \quad (\text{A.5})$$

$$\begin{aligned} \tau_{\mu\nu} &= - \left(\perp_\mu^\alpha \perp_\nu^\alpha - \frac{1}{3} \perp_{\mu\nu} \perp^{\alpha\beta} \right) \nabla_\alpha \nabla_\beta \pi = \\ &= - \left(\overline{\nabla}_\nu \overline{\nabla}_\mu - \frac{1}{3} \perp_{\mu\nu} \Delta \right) \pi + \frac{d\pi}{d\tau} \sigma_{\mu\nu} + u_\mu \sigma_\nu^\beta \overline{\nabla}_\beta \pi + \frac{1}{3} \theta u_\mu \overline{\nabla}_\nu \pi, \end{aligned} \quad (\text{A.6})$$

where q denotes a potential for the energy flow while π is the anisotropic stress, which plays the role of a potential for the viscous-stress tensor. They are scalar functions of the covariantly defined variables ϕ , m (the value of “velocity” of the scalar) and potentially other scalars defined on the spatial hypersurface, say θ or ${}^{(3)}R$, the intrinsic curvature of the three-slice (see eq. (A.11)). The symbol $\Delta \equiv \perp^{\mu\nu} \overline{\nabla}_\mu \overline{\nabla}_\nu$ is the Laplacian on the spatial hypersurface.

Admittedly, such a simplification may not always be quite possible: the model’s viscous-stress tensor could contain some uncontracted function of the curvature tensor ${}^{(3)}R_{\mu\nu}$. One may need to deal with these intrinsically non-scalar terms separately. The validity of such a decomposition should be confirmed on a model-by-model basis.

We should note at this stage that a covariant spatial derivative $\overline{\nabla}_\mu$ acting on a function of only ϕ vanishes (since $\overline{\nabla}_\mu f(\phi) = f_\phi \perp_\mu^\nu \nabla_\nu \phi = -m f_\phi \perp_\mu^\nu u_\nu = 0$, i.e. since the gradient of ϕ defines the time direction). In many models, such as $f(R)$ gravity or the non-minimally coupled *k-essence* model discussed in section 3, the anisotropic stress is only a function of ϕ and therefore the viscous stress tensor reduces to

$$\tau_{\mu\nu} = \frac{d\pi(\phi)}{d\tau} \sigma_{\mu\nu}, \quad (\text{A.7})$$

and the function $d\pi/d\tau$ plays the role of shear viscosity. Note that depending on the direction of evolution of ϕ , this shear viscosity can be of either sign. This is a hint that we are not dealing with a usual fluid.

Assuming eqs (A.5) and (A.6), we can re-express the EMT conservation equations (A.3) and (A.4) for the case where the substitution of (A.5) and (A.6) is valid

$$-u^\nu \nabla^\mu T_{\mu\nu} = \frac{d\mathcal{E}}{d\tau} + \theta(\mathcal{E} + \mathcal{P}) + 2a^\mu \overline{\nabla}_\mu q + \Delta q + \frac{d\pi}{d\tau} \sigma_{\mu\nu} \sigma^{\mu\nu} - \sigma^{\mu\nu} \overline{\nabla}_\mu \overline{\nabla}_\nu \pi, \quad (\text{A.8})$$

$$\perp_\lambda^\nu \nabla^\mu T_{\mu\nu} = a_\lambda \left(\mathcal{E} + \mathcal{P} + \frac{dq}{d\tau} \right) + \overline{\nabla}_\lambda \left(\mathcal{P} + \frac{dq}{d\tau} \right) + \theta \overline{\nabla}_\lambda q + \perp_{\lambda\nu} \nabla_\mu \tau^{\mu\nu}. \quad (\text{A.9})$$

The gradient of viscous stress can be expressed as

$$\begin{aligned} \perp_{\lambda\nu} \nabla_\mu \tau^{\mu\nu} &= \frac{d\pi}{d\tau} R_{\mu\nu} \perp_\lambda^\nu u^\mu + \frac{2}{3} \dot{\pi} \overline{\nabla}_\lambda \theta + \frac{d\pi}{d\tau} a_\mu \sigma_\lambda^\mu + \sigma_\lambda^\mu \overline{\nabla}_\mu \dot{\pi} - \\ &\quad - a^\alpha \overline{\nabla}_\lambda \overline{\nabla}_\alpha \pi - \frac{2}{3} \overline{\nabla}_\mu \Delta \pi + \frac{1}{3} a_\lambda \Delta \pi - {}^{(3)}R_\lambda^\mu \overline{\nabla}_\mu \pi, \end{aligned} \quad (\text{A.10})$$

with the intrinsic curvature of the spatial hypersurface obtained through the Gauss-Codazzi equation

$${}^{(3)}R_{\mu\nu} = \sigma_{\mu\alpha} \sigma_\nu^\alpha - \frac{1}{3} \theta \sigma_{\mu\nu} - \frac{2}{9} \theta^2 \perp_{\mu\nu} + R_{\rho\sigma\alpha\beta} \perp^{\alpha\rho} \perp_\mu^\beta \perp_\nu^\sigma. \quad (\text{A.11})$$

If eq. (A.7) is satisfied then eq. (A.10) simplifies further,

$$\perp_{\lambda\nu} \nabla_\mu \tau^{\mu\nu} = \frac{d\pi}{d\tau} \left(R_{\mu\nu} \perp_\lambda^\nu u^\mu + \frac{2}{3} \nabla_\lambda \theta \right). \quad (\text{A.12})$$

The advantage of the fully generally covariant decomposition of the EMT conservation presented in eqs (A.8) and (A.9) is that it is exact under the circumstances where the substitution of (A.5) and (A.6) is valid. As we demonstrate below, obtaining the linear perturbations equations for a background flat FLRW metric is very simple. However, starting from these covariant equations one can also easily derive the perturbation equations for any other background, for example, a cosmology where the background shear is non-vanishing (Bianchi) or there is some sort of (potentially non-isotropic) spatial curvature. In particular, it is easy to see from eq. (A.10) that for models where the anisotropic stress is more complicated than in eq. (A.7) there is a non-trivial coupling to background spatial curvature which will contribute to the evolution of perturbations.¹⁷

We should note that the equation for the conservation of the EMT for a scalar field is proportional to the equation of motion for the scalar. This must be so, since the scalar is a single degree of freedom and providing a second evolution equation would overconstrain it. This means that for all scalar-field Lagrangians, eq. (A.8) is equivalent to the equation of motion while eq. (A.9) vanishes identically on all configurations. The momentum conservation equation can in fact be shown to relate the parts of the pressure and energy density for the scalar, providing a sort of equation of state or, more correctly, it is the equilibrium Euler relation for the scalar field (see e.g. refs [45, 58]).

B Choice of frames and boosts

We have performed the decomposition of the EMT in what we have called the scalar frame, with the result that the fluid variables we've obtained (energy density, pressure, etc.) have the values which would have been seen by an observer moving with velocity u^μ defined in eq. (2.2). A different observer will in general observe a different energy density, etc. (see also the discussion in ref. [107]).

The *k-essence* class of models has EMTs of perfect-fluid form. The energy-flow q^μ in the scalar frame vanishes and thus the scalar frame is also the Landau-Lifshitz frame (or rest frame). This makes the choice of frame for the analysis very simple: the scalar frame is the natural one. However, in general scalar-field models (for example, see ref. [58] or section 3), the energy flow q_μ in the scalar frame is non-vanishing. In these circumstances, the LL frame is distinct from the scalar frame and one needs to choose. Let us compare the physics in the two frames.

The two frames are related by a (space-dependent) Lorentz boost. However, in general there isn't a simple transformation between the two frames. For the purposes of cosmology, q_μ and $\tau_{\mu\nu}$ in the background vanish because of the symmetry of the FLRW metric, so we are able to make a perturbative statement about this transformation.

The Lorentz boost we require is a transformation from the orthogonal vector pair (u^μ, q^μ) to the orthogonal pair (U^μ, Q^μ) defined by the linear transformation

$$\begin{pmatrix} U^\mu \\ \hat{Q}^\mu \end{pmatrix} = \begin{pmatrix} \cosh \alpha & \sinh \alpha \\ \sinh \alpha & \cosh \alpha \end{pmatrix} \begin{pmatrix} u^\mu \\ \hat{q}^\mu \end{pmatrix}, \quad (\text{B.1})$$

¹⁷For example, if the action for the scalar contains a term $G_{\mu\nu} \nabla^\mu \phi \nabla^\nu \phi$ [106].

where \hat{q}^μ is the unit vector in the direction q^μ and the rapidity α is defined as

$$\alpha^2 \equiv \frac{q^\alpha q_\alpha}{\mathcal{E} + \mathcal{P}}, \quad (\text{B.2})$$

or

$$U_\mu \simeq u_\mu + \frac{q_\mu}{\mathcal{E} + \mathcal{P}}, \quad (\text{B.3})$$

to first order in α . The frame comoving with velocity U^μ is now the LL (“rest”) frame. In particular, the EMT decomposed in this new frame to leading order in α can be written

$$\begin{aligned} T_{\mu\nu} \simeq & \left(\mathcal{E} - \frac{2q_\alpha q^\alpha}{\mathcal{E} + \mathcal{P}} \right) U_\mu U_\nu + \left(\mathcal{P} - \frac{2q_\alpha q^\alpha}{3(\mathcal{E} + \mathcal{P})} \right) \perp_{\mu\nu}^U + \\ & + \left(\tau_{\mu\nu} - \frac{2 \left(q_\mu q_\nu - \frac{1}{3} q_\alpha q^\alpha \perp_{\mu\nu}^U \right)}{\mathcal{E} + \mathcal{P}} \right). \end{aligned} \quad (\text{B.4})$$

with $\perp_{\mu\nu}^U \equiv g_{\mu\nu} + U_\mu U_\nu$. This implies that up to first order in the rapidity α , an observer comoving with velocity U^μ will observe the same energy density, pressure and viscous stress as one comoving with u^μ : we have

$$T_{\mu\nu} \simeq \mathcal{E} U_\mu U_\nu + \mathcal{P} \perp_{\mu\nu}^U + \tau_{\mu\nu}. \quad (\text{B.5})$$

Just as required, the energy flow in this frame vanishes. However, the decomposition (B.4) shows that the observed LL fluid variables are in fact modified with respect to the scalar frame already at the second order in α .

One should note at this point that the above is not a gauge-dependent statement. Frequently the question of gauge and frame choice are conflated in cosmology. They are in principle very different. Gauges are a choice of the form of the metric perturbations which are all equivalent as a result of diffeomorphism invariance of general relativity. They do not impact observables. Since the Newtonian potentials Φ and Ψ are gauge invariant and are equal to the potentials ϕ_L and ψ_L in the longitudinal gauge, one should consider the perturbed equations (2.18) and (2.19) to be gauge-invariant themselves, provided that each of the quantities is interpreted as the gauge-invariant variable with appropriate transformations given different gauge choices. The choice of frame, on the other hand, is a statement about observables. As can clearly be seen in eq. (B.4), the quantity that the observer moving with U^μ will call the energy density is a scalar, but it is a different scalar to the energy density seen by u^μ .

Turning to first-order cosmological perturbation theory, we can find the spatial velocity divergence for U^μ corresponding to (2.20),

$$\Theta_{\text{LL}} \equiv i k_i \delta U^i \simeq \Theta_X - \frac{k^2 \delta q}{a^2 (\mathcal{E} + \mathcal{P})} + \frac{\dot{q}}{\mathcal{E} + \mathcal{P}} \Theta_X, \quad (\text{B.6})$$

and therefore we can rewrite the conservation equations (2.18) and (2.19) in the Landau-Lifshitz frame,

$$\begin{aligned} \delta \dot{\mathcal{E}}_{\text{LL}} + 3H (\delta \mathcal{E}_{\text{LL}} + \delta \mathcal{P}_{\text{LL}}) + (\mathcal{E} + \mathcal{P}) (\Theta_{\text{LL}} + 3\dot{\Phi}) &= 0, \\ (\mathcal{E} + \mathcal{P}) \left(\dot{\Theta}_{\text{LL}} + 2H \Theta_{\text{LL}} - \frac{k^2}{a^2} \Psi \right) - \frac{k^2}{a^2} \delta \mathcal{P}_{\text{LL}} + \Theta_{\text{LL}} \dot{\mathcal{P}} - \frac{2}{3} \frac{k^4}{a^4} \delta \pi_{\text{LL}} &= 0. \end{aligned} \quad (\text{B.7})$$

where the subscript LL on the fluid variables signifies that these are the quantities observed in the LL frame and just happen to be equal to the scalar-frame equivalents up to first order in α . Unsurprisingly all that has happened is that the terms containing q have disappeared: the form of these equations must be invariant under Lorentz boosts.

One could at this stage walk away having decided that one should always perform cosmological perturbation theory in the Landau-Lifshitz (rest) frame since it appears computationally simpler as a result of having eliminated the additional variables q . However, this is dangerous for two reasons:

- The velocity field U^μ is now no longer vorticity-free. The scalar-frame velocity field was a (normalised) derivative of a scalar field and thus the antisymmetric rotation (twist) tensor vanished. This is no longer true for U^μ and therefore the scalar-field fluid can carry vector perturbations in this frame—albeit only at higher orders in perturbations. One may not be particularly worried about this since at higher orders all the perturbation types mix anyway, but it does provide additional complication, especially with regard to choosing a good Cauchy surface. One would also need to add the rotation tensor terms to eqs (A.3) and (A.4) and would not be able to perform the decomposition leading to (A.8) and (A.9).
- Possibly more importantly, eq. (B.2) shows that the rapidity α required to boost to the LL frame diverges when the equation of state of the fluid approaches $w = -1$. The physical origin of this is that the vanishing enthalpy of the vacuum cannot compensate for the energy flow q_μ . This means that as the fluid approaches the vacuum equation of state, the perturbative relations between the fluid variables in the LL and any other frame breaks down: in the LL frame, the energy-density functional form itself contains corrections (the leading one can be seen in (B.4)), which are order one close to $w = -1$. This also means that order-one vorticity would be observed in the LL frame.

The last point implies that if there is any danger of having an equation of state close to that of vacuum, one should *not* decompose the EMT in the LL frame and attempt to evolve these hydrodynamic variables. On the other hand, any *other* choice of frame is in fact equivalent up to first order in the difference in velocities and for the purpose of linear perturbation theory one can directly use results from one frame and apply them to others.

One could of course claim that such a setup is not possible: after all one calculates the hydrodynamical EMT by averaging quantities in the rest frame of the particles making up the fluid in the first place. Why should it be possible to produce an EMT which one could not put back in the rest-frame? As we argue in section 2.4, general scalar-field theories do not have EMTs which behave as in hydrodynamics and indeed have been shown to be able to produce a fluid vacuum configuration than nonetheless carries momentum in the spatial direction (e.g. ref. [58])

It is worth noting here that the practice assigning the meaning of e.g. the energy density to components of the EMT can also be understood in the framework of the decomposition (2.14). First, one should realise that it is *impossible* to find a frame where the definition of energy density is *exactly* $-T^0_0$ given an arbitrary metric. The best we can do is to pick the time direction to be that of the coordinate time of the background, i.e.

$$u^\mu_{\text{com}} = \sqrt{-g^{00}} \delta^\mu_0, \quad (\text{B.8})$$

where the square root of the metric is necessary for the appropriate normalisation of the velocity. In this frame we can then obtain the following decomposition of the EMT:

$$\begin{aligned}\mathcal{E}_{\text{com}} &= -g^{00}T_{00} = -T_0^0 + g^{0i}T_{i0}, \\ \mathcal{P}_{\text{com}} &= \frac{1}{3} \left(T_\mu^\mu - \mathcal{E} \right), \\ q_\lambda^{\text{com}} &= \delta_\lambda^i \sqrt{-g^{00}} (T_{0i} + g_{0i}\mathcal{E}), \\ \tau_{\mu\nu}^{\text{com}} &= \delta_\mu^i \delta_\nu^j \left(T_{ij} - \mathcal{P}g_{ij} - g^{00}(\mathcal{E} - \mathcal{P})g_{0i}g_{0j} - g^{00}(T_{0i}g_{0j} + T_{0j}g_{0i}) \right).\end{aligned}\tag{B.9}$$

This then matches the components approach, but only provided that the $(0i)$ metric elements vanish. Even in this case there is an additional metric factor multiplying q_λ . However, given scalar perturbation of FLRW in the Newtonian gauge and neglecting second-order corrections, this decomposition yields the same energy density, pressure and viscous stress as the decomposition in the scalar frame.

Another aspect worth mentioning is what happens in the case of an external EMT. For example, if we consider the scalar field to model dark energy, dark matter would be an additional external perfect fluid. The external EMT will contain its own rest frame with velocity, say, v^μ

$$T_{\mu\nu}^{\text{ext}} = \rho_{\text{ext}}v_\mu v_\nu + p_{\text{ext}} \perp_{\mu\nu}^V, \quad \perp_{\mu\nu}^V \equiv g_{\mu\nu} + v_\mu v_\nu.\tag{B.10}$$

However, we have to pick just one frame in which to do the decomposition, and it is convenient to make that the scalar frame. This means that the fluid variables describing the dark matter are not those observed in its own rest frame, but appropriately transformed quantities. For example, the external pressure observed in the frame u^μ would then be

$$p_{\text{ext}}^U = \frac{1}{3}p_{\text{ext}} \left(2 + (u \cdot v)^2 \right) - \frac{1}{3}\rho_{\text{ext}} \left(1 - (u \cdot v)^2 \right).\tag{B.11}$$

The various properties of the external perfect fluid are defined in its own rest frame (just as they are for *k-essence*, for example). Thus in general one needs to keep track of how the external matter appears in the frame in which we are decomposing and perform the perturbations. For example, eq. (B.11) show that external dust, which has an equation of state $w_{\text{ext}} = 0$ in its own frame, would have a non-vanishing pressure in the scalar frame.

However, just as in the case of the perturbations of the scalar, if both the fluids share the FLRW background, the corrections to energy density, pressure and viscous stress arising from the difference in the rest-frames appear only at second order, thus for example the energy density perturbations $\delta\rho_{\text{ext}}$ in the external rest-frame is going to be the same as in the scalar frame and we can drop the superscript U used in eq. (B.11) to differentiate the two. At linear order the only difference will be in the energy flow: since the two fluids do not share the same rest frame, the external matter will have a non-vanishing velocity from the point of view of the scalar:

$$\delta q_\lambda^{\text{ext}} \equiv T_{\mu\nu}^{\text{ext}} u^\mu \perp_\lambda^\nu \simeq (\rho_{\text{ext}} + p_{\text{ext}}) \delta_\lambda^i (\delta v_i - \delta u_i).\tag{B.12}$$

or if we take the divergence,

$$ik_i \delta q_{\text{ext}}^i = (\rho_{\text{ext}} + p_{\text{ext}}) (\Theta_{\text{ext}} - \Theta),\tag{B.13}$$

where $\Theta_{\text{ext}} = \partial_i \delta v^i$ is the analogously defined spatial divergence of the external matter velocity perturbation.

To conclude: the choice of frame is in general important when one is worried about higher orders in cosmological perturbation theory. If we only concern ourselves with linear perturbations, then any frame where the observers while unperturbed are comoving with the FLRW background observers will give the same results. However, for scalar-field models with second-derivatives in the EMT, the Landau-Lifshitz (rest) frame is singular when $w = -1$ and perturbation theory loses validity as the fluid approaches the vacuum-energy equation of state. Thus, somewhat counter intuitively to expectations, one should use any frame but the rest frame to study linear perturbations in cosmology.

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